SCM Lecture 7 - teaching notes

Jeanne Wilson

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1 Poisson Distribution

What is a Poisson process? Often talk about it as a counting experiment - counting individual occurrences of an "event".

- Poisson distribution is a discrete probability distribution (can only have certain values - whole numbers of events)
- Probability of a number of events occurring in a fixed period of time
- Events occur with a known average rate but each event is independent of last. (ie. occurring randomly, not exactly every second)

$$P_\lambda(n, \lambda) = \frac{\lambda^n \mathrm{e}^{-\lambda}}{n!}$$

That is the probability of n occurrences of an event given that the average number is λ. e is the natural base logarithm = 2.71828...

For example: count number of radioactive decays. Start with $N = 10^{20}$ atoms, the probability of an atom to decay within 1 second is $p = 2 \times 10^{-20}$

What is the average number of decays per second?

$$\lambda = Np = 2$$

So if λ is the average, how many decays are we likely to see? We need an idea of the spread of the distribution really.

Remember back to last lecture - we talked about the binomial distribution that applies when there are only 2 outcomes - true and false. That sort of applies here right - the decay either happens or it doesn't.

So the Poisson distribution is a special case of the Binomial distribution. We are interested in decays in some interval, T. Consider splitting that up into N chunks, each with a small probability of 1 decay from the entire radioactive sample, p. Probability of no decays in a given chunk = $(1-p)$. So the probability of n decays in our interval T is

$$P_{N,p}(n) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n} \tag{1}$$

What happens to the factor at the front if N is large and p is small? (and therefore Np is finite).
$\frac{N!}{(N-n)!} = \frac{N.(N-1).(N-2)....3.2.1}{(N-n).(N-n-1).(N-n-2)...3.2.1} = N.(N-1)...(N-n+1) \approx N^n$
ie. The top cancels with the bottom down to $(N-n+1)$ but if N is large with respect to n then that can be approximated to N^n giving us

$$\begin{aligned} P_{N,p}(n) &\approx \frac{N^n}{n!} p^n (1-p)^{N-n} \\ &\approx \frac{N^n}{n!} p^n \frac{(1-p)^N}{(1-p)^n} \\ &\approx \frac{(Np)^n}{n!} (1-p)^N \end{aligned}$$

Note - if p is small and n is small with respect to N then $(1-p)^n$ is approximately 1.
Substitute $\lambda = Np$:

$$P_{N,p}(n) \approx \frac{\lambda^n}{n!} (1-p)^{\lambda/p}$$

Now we take the limit where $p \to 0$

$$\begin{aligned} \lim_{p \to 0} (1-p)^{1/p} &= \lim_{x \to 0} \left(1 - \frac{1}{x}\right)^x = e^{-1} \\ \lim_{p \to 0} P_{N,p}(n) &= \frac{\lambda^n e^{-\lambda}}{n!} \end{aligned}$$

Which brings us back to the Poisson probability distribution.

Now remember the expectation value for the binomial:
$E[P_{N,p}(n)] = Np$ so for the Poisson the expectation is $E[P_\lambda(n)] = \lambda$
And the variance for the binomial:
$\text{var}[P_{N,p}(n)] = Np(p-1)$ but when $p \to 0$, $(1-p) \to 1$ so for the Poisson the expectation is $\text{var}[P_\lambda(n)] = \lambda$

Note that is the variance, so the likely spread in the number of decays seen is $\sigma = \sqrt{\lambda}$
so the spread of our decay example distribution is $\sqrt{2}$.

Look at some distributions

The distribution becomes more symmetric as λ increases. It only depends on λ so doesn't matter if its N or p that increases. What is the fractional error?

$$\frac{\sigma}{\lambda} = \frac{\sqrt{\lambda}}{\lambda} = \frac{1}{\sqrt{\lambda}}$$

so as λ increases, the fractional error decreases.

1.1 Examples

- Radioactive decay
- Noise. I am using a photomultiplier tube with a dark noise rate of 5 Hz. Photons impinging on the PMT result in electrical pulses, however, sometimes we get spurious pulses due to other excitations when no photons are present. This means that there are 5 pulses a second due to noise not photons. I see a signal of 10 photons in a second - is this likely to be real light or just a fluctuation in the noise?
  $\lambda = 5, P(10) = \frac{\lambda^n e^{-\lambda}}{n!} = \frac{5^{10}.e^{-5}}{10!} = 0.018 = 1.8\%$
  ie. its less than 2% chance that this is a fluctuation in the noise.

2 The Uniform Distribution

So far we have been considering distributions of discrete events: integer numbers of events etc. but the event space could also be continuous - ie. consisting of an infinite number of real values.

$$\begin{aligned} P(x < X < x + dx) &= f(x)dx \\ P(a < X < b) &= \int_a^b f(x)dx \\ \int_{-\infty}^{+\infty} f(x)dx &= 1 \end{aligned} \tag{2}$$

The uniform distribution is a continuous distribution of equal probabilities.

$$f(x) = K = \text{constant}$$

It must be normalised over the range for which it is valid. eg. $R = a$—b

$$\begin{aligned} \int_{-\infty}^{+\infty} f(x)dx &= 1 \\ \int_a^b K dx &= K(b-a) = 1 \\ K = \frac{1}{b-a} & \\ f(x) = \frac{1}{b-a} & \end{aligned} \tag{3}$$

What are the expectation and variance of the uniform distribution?
We will use the standard technique for evaluating an expectation value:

$$E[x] = \int_a^b x.f(x)dx = \int_a^b x.\frac{1}{b-a}dx$$

$$
\begin{aligned}
E[x] &= \frac{1}{b-a}\frac{x^2}{2}\,|_a^b = \frac{1}{2(b-a)}(b^2-a^2) = \frac{1}{2(b-a)}(b-a)(b+a) \\
E[x] &= \frac{1}{2}(b+a)
\end{aligned}
$$

and for the variance

$$
\begin{aligned}
\mathrm{var}[x] &= \int_a^b (x-E[x])^2 f(x)dx \\
\mathrm{var}[x] &= \int_a^b (x-\frac{1}{2}(b+a))^2 \frac{1}{b-a}dx \\
&= \frac{1}{b-a}\int_a^b (x^2 - x(b+a) + \frac{1}{4}(b+a)^2)dx \\
&= \frac{1}{b-a}\left[\frac{x^3}{3} - \frac{x^2}{2}(b+a) + \frac{x}{4}(b+a)^2\right]_a^b \\
&= \ldots \\
&= \frac{1}{12}(b-a)^2
\end{aligned}
$$

3 The Gaussian Distribution

Also called

- Normal distribution
- Bell curve

Most important probability distribution as it occurs naturally for many processes.
Can also be derived as the large N limit of the Binomial distribution.

$$
G(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{4}
$$

So this distribution depends on two parameters, μ and σ. Lets see what we can work out about the distribution from this equation.

- *For what x value does this distribution take its maximum?*

  $x = \mu$ (The maximum value of e^{-A} is when $A = 0$).

- Now note that the x dependence comes in within this squared term. *What does that tell us?*

  The distribution must be symmetric - $\pm(x-\mu)$ give the same value so that maximum value must be the expectation value. Its the center of the distribution and the most probable value. We could also prove that by doing.

$$
E[X] = \int xG(x)dx = \mu \tag{5}
$$

- *And what is the amplitude at the maximum?*
  $\frac{1}{\sqrt{2\pi}\sigma}$ $(e^0 = 1)$

- The $\frac{1}{\sqrt{2\pi}}$ is a normalisation factor.

- We could also have obtained the maximum by differentiating: $\frac{dG(x)}{dx}$. *What does the differential of a distribution tell us?*

  The gradient - when the gradient = 0 we are at a turning point - either a maximum or a minimum.

- *What does the double differential tell us:* $\frac{d^2G(x)}{dx^2}$ *?*
  This tells us stationary points - in this case the point of maximum gradient.
$$\frac{d^2G(x)}{dx^2} = 0 \rightarrow x = \mu \pm \sigma \tag{6}$$
  So the turning point of the distribution is at the mean $\pm\sigma$.

- *What is the variance of the distribution?*
$$\text{var}[X] = E[(X - E[X])^2] = \int (x-\mu)^2 G(x) dx = \sigma^2 \tag{7}$$
  So the variance is σ^2, the standard deviation is σ.

- *What is the Full Width at Half Maximum?*
  We know the maximum amplitude is $\frac{1}{\sqrt{2\pi}\sigma}$ so we need to know at what x values, $G(x)$ is equal to half of this
$$\begin{aligned}
G(x) &= \frac{1}{2\sqrt{2\pi}\sigma} \\
\frac{1}{2\sqrt{2\pi}\sigma} &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \\
\frac{1}{2} &= e^{-\frac{(x-\mu)^2}{2\sigma^2}} \\
\ln\frac{1}{2} &= -\frac{(x-\mu)^2}{2\sigma^2} \\
-\ln 2 &= -\frac{(x-\mu)^2}{2\sigma^2} \\
2\sigma^2 \ln 2 &= (x-\mu)^2 \\
\pm\sigma\sqrt{2\ln 2} &= (x-\mu) \\
x &= \mu \pm \sigma\sqrt{2\ln 2} \\
\text{FWHM} &= 2\sqrt{2\ln 2}\sigma = 2.35482...\sigma
\end{aligned}$$

- Limit as $x \to \pm\infty = 0$

- Like any probability distribution:

$$\int_{-\infty}^{\infty} G(x) = 1 \tag{8}$$

Now look at slide showing all these things