# SCM Lecture 5 - teaching notes

Jeanne Wilson

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## 1 Some definitions

In the previous lectures we talked a lot about errors and uncertainty. Hopefully we have impressed on you that virtually everything has some uncertainty associated with it. Quite often that uncertainty is so small we really don't care, but physics is a precision science and we often do care. Therefore, probability is a very important concept in physics, especially quantum physics.. eg:

- I measure the energy of 100 gammas emitted by a radioactive sample and find the mean energy is $500 \pm 10\,\mathrm{MeV}$. I can't say for certain what the next energy I will measure is but there is 68% probability that it will be in the range 490–510 MeV. And I can't give you the definite true gamma energy, but I can be 68% sure that it is within $\frac{10}{\sqrt{100}}$ of the mean - *i.e.* $100 \pm 1\,\mathrm{MeV}$.

Express probability as a dimensionless number between 0 (impossible) and 1 (certain).

I will use the classic probability example of throwing a dice. Some events are certain, for example, I'm sure I'll throw a number between 1–6. Some are impossible - I won't throw a 0. And some are random - there is a $\frac{1}{6}$ chance that I will throw a 3. Here are some definitions:

- **Sample Space** = the set of all possible outcomes of the experiment.
- **Event** = a subset of the sample space. *i.e.* a set of outcomes.
- **Elementary Event** = a single point in sample space.

Suppose my "measurement" is the two values I get from two throws of the dice (I don't care about the order)

- **Sample Space** = 11,12,13,14,15,16, 22, 23, 24, 25, 26, 33, 34, 35, 36, 44, 45, 46, 55, 56, 66
- **Event** = throw a double = 11, 22, 33, 44, 55, 66
- **Elementary Event** = 13

We can combine events in various ways - probability algebra:

- **Union:** $\mathbf{A+B = A \cup B}$ the set of elementary events that belong to A or B -*OR*
- **Product:** $\mathbf{A.B = A \cap B}$ the set of elementary events that belong to A and B -*AND*
- **Complementarity** $\bar{\mathbf{A}}$ the set of elementary events that do not belong to A -*NOT*

### 1.1 Example

Throw a dice.

- Event $\mathbf{A}$ = Throw $\geq 3$ = (3, 4, 5, 6): 4 elementary events
- Event $\mathbf{B}$ = Throw even number = (2, 4, 6): 3 elementary events
- $\mathbf{A \cup B}$ = (2, 3, 4, 5, 6) : 5 elementary events
- $\mathbf{A \cap B}$ = (4, 6) : 2 elementary events
- $\bar{\mathbf{A}}$ = (1, 2) : 2 elementary events
- $\bar{\mathbf{B}}$ = (1, 3, 5) : 3 elementary events
- $\bar{\mathbf{A}} \cap \mathbf{B}$ = (2) : 1 elementary event
- $\mathbf{A} \bar{\cap} \mathbf{B}$ = (1, 2, 3, 5) : 4 elementary events

## 2 Probability Algebra

- **Mutually exclusive events:** $\mathbf{A \cap B = 0}$
- $\mathbf{P(\bar{A}) = 1 - P(A)}$
- $\mathbf{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$

*See slide* As shown on this slide - if we take the area in A and add the area in B we are double counting the overlap region, D, which is A and B so we take that away.

### 2.1 Conditional probability

We can ask what the probability is of a given outcome, given that something else has happened. For example, throw a dice twice - there are 36 possible outcomes, *i.e.* 36 elementary events. Define

- **Event A** = sum of two dice is even. $P(A) = \frac{18}{36}$
- **Event B** = sum of two dice is 8. $P(B) = \frac{5}{36}$

*See slide* There are 18 elementary events in B and all of the events in event A are in event B. We can say A is a subset of B.
So what is the probability that you get an 8 given that it is even? We write this as $P(A|B) = \frac{5}{18}$

In general we say

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{1}$$

*See slide* Note we choose events in the region $r$ but we give the probability as a fraction of the elementary events in B: $k$, not the total number of elementary events: $n$. Basically, we redefine our sample space as event B.

## 3 Probability Interpretation

So that was the easy bit - defining the sample space. Lets now think about the definition of probability. This is a bit more complicated, like religion. there are two broad categories of probability interpretations, whose adherents possess different (and sometimes conflicting) views about the fundamental nature of probability.

- **Bayesian** : evidential probability
  Classical interpretation (Laplace):
  $P(A) = \frac{n}{N}$ where n is the number of cases of n and N is the total number of possible outcomes. This works for something like throwing a dice. If A is the probability of getting a 3, $n = 1$ and $N = 6$. However, the problem with this definition is that there isn't always a finite number of outcomes, $N$. It also assumes that all events are equally likely. The concept of "likely" is already in the definition.
  Probability, for a Bayesian, is a way to represent an individual's degree of belief in a statement, or an objective degree of rational belief, given the evidence.

- **Frequentists** : physical probability
  In this approach you consider the "frequency" of outcomes. $F(A) = \frac{n}{M}$ where M is the number of repetitions of the test/measurement. If an infinite number of tests are made you get the probabilty:
  $P(A) = \lim_{M \to \inf} F(A)$
  This approach doesn't require equally likely events, but it does require an infinite number of repetitions to define a probability. If we acknowledge the fact that we only can measure a probability with some error of measurement attached, we still get into problems as the error of measurement can only be expressed as a probability, the very concept we are trying to define.
  Frequentists consider probability to be the relative frequency "in the long run" of outcomes.

## 4 Probability Distributions

Consider a sample space consisting of $X_i$ events, each with probability $P(X_i)$. A plot of $P(X_i)$ against $X_i$ is called a Probability histogram or probability distribution function (PDF). You can make a PDF by histogramming your data and normalising that histogram.

For example - lets consider dates of birth in the year. It makes sense to divide our histogram X-axis into 12 for the months (lets ignore the fact that they are slightly different widths due to the different number of days in a month). So we ask 100 people and we get a graph like this (*see slide*) and we normalise it by dividing the contents of each bin by 100 and get this (*see slide*). Reading off the y-axis for a given bin gives us the probability that somebody's birthday (out of the people we surveyed) is in that month. For the frequentist approach we would have to ask an infinite number of people...