SCM Lecture 4 - teaching notes

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1 Example 1 Resistors

Two resistors are measured to have
$R_1 = (2.00 \pm 0.01)k\Omega$
$R_2 = (4.00 \pm 0.01)k\Omega$
What is the ratio of $\frac{R_1}{R_2}$?
Now in this case we think the uncertainties are correlated - if we put our resistors into the same measuring circuit, errors are likely to be due to additional resistance we haven't taken into account or biases in the meters so its likely if we over-estimated one resistance, we over-estimate the other.

$G = \frac{R_1}{R_2} = 0.465 \pm \Delta G$

Units? - none its a ratio

OK - lets do it the long way, which you can always fall back on if you don't remember the error formula. Differentiate using the chain rule

$$\begin{aligned}\Delta G &= \left|\frac{\delta G}{\delta R_1}\right| \Delta R_1 + \left|\frac{\delta G}{\delta R_2}\right| \Delta R_2 \\ \Delta G &= \frac{1}{R_2}\Delta R_1 + \frac{R_1}{R_2^2}\Delta R_2\end{aligned}$$

divide through by $G = \frac{R_1}{R_2}$:

$$\begin{aligned}\frac{\Delta G}{G} &= \frac{R_2}{R_1}\frac{1}{R_2}\Delta R_1 + \frac{R_2}{R_1}\frac{R_1}{R_2^2}\Delta R_2 \\ \frac{\Delta G}{G} &= \frac{\Delta R_1}{R_1} + \frac{\Delta R_2}{R_2}\end{aligned}$$

Precision in R_1 is 0.5%, in R_2 is 0.25% so precision in ratio is 0.75%, $G = 0.5 \pm 0.00375$. So how should we quote that?

$$G = 0.5000 \pm 0.0038.$$

1.1 Resistors in series

Place these two resistors in series $R_T = R1 + R_2$. Now lets assume that their uncertainties are *uncorrelated*. What is the error on the total resistance?

$$\Delta R_T = \sqrt{\Delta R_1^2 + \Delta R_2^2}$$
$$R_T = 6.00 \pm \sqrt{0.01^2 + 0.01^2} = 6.00 \pm 0.014 k\Omega$$

1.2 Resistors in parallel

Now what if we place them in parallel? $\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2}$ or $R_T = \frac{R_1 R_2}{R_1 + R_2}$
What is the error on R_T now - we have both multiplicative and additional errors.

$$\frac{\Delta R_T}{R_T} = \sqrt{\left(\frac{\Delta R_1^2 + \Delta R_2^2}{(R_1+R_2)^2} + \left(\frac{\Delta R_1}{R_1}\right)^2 + \left(\frac{\Delta R_2}{R_2}\right)^2\right)}$$
$$\frac{\Delta R_T}{R_T} = \sqrt{\left(\left(\frac{0.014}{6.00}\right)^2 + \left(\frac{0.01}{2.00}\right)^2 + \left(\frac{0.01}{4.00}\right)\right)^2}$$
$$R_T = 1.333 \pm 0.008 k\Omega$$

2 Example 2 Time period of a pendulum

Measuring the oscillation time with a stop watch. There is an uncertainty due to your reaction time so measure 10 oscillations, $\Theta = 10T$, to reduce this.

Measure $\Theta = (16.20 \pm \Delta\Theta)$s.
$\Theta = T_{\text{stop}} - T_{\text{start}}$
The time we measure is the true time plus reaction time, R. We can measure our reaction time repeatedly to get an idea of the mean and standard deviation:
$R = \bar{R} \pm \Delta R = (0.3 \pm 0.2)$s
so $T_{\text{start}} = 0 + \bar{R} \pm \Delta R$ and $T_{\text{stop}} = T_s + \bar{R} \pm \Delta R$
Subtracting these cancels out $\bar{R}$ but leaves us with two ΔR values.

- Are these correlated? No - there is no reason why we should be consistently slow or consistently fast in the two measurements, it should be random so that means we should be adding things in quadrature.
- Do we work with relative or absolute uncertainties? We're just subtracting so absolute
  $\Delta\Theta = \sqrt{(\Delta R)^2 + (\Delta R)^2} = \sqrt{2(0.2)^2} = 0.283$s
- OK so we have the error on Θ - how does that convert into error on period, T?
  $T = \frac{\Theta}{10}$, there is no doubt about the 10 so the relative uncertainty, or precision, in T should be the same as the relative uncertainty in Θ:
  $\frac{\Delta T}{T} = \frac{\Delta\Theta}{\Theta} \quad \rightarrow \quad \Delta T = \frac{1}{10}\Delta\Theta = \frac{0.283}{10}$

$$T = (1.62 \pm 0.03)\text{s}$$

3 Example 3 - Pendulum to find g

$$T = 2\pi\sqrt{\frac{l}{g}} \rightarrow g = 4\pi^2 \frac{l}{T^2} \tag{1}$$

$T = 1.62 \pm 0.03$s, measure $l = (65.1 \pm 0.1)$cm. What is the error on g?
Are the errors likely to be correlated? - No so we need to add in quadrature. Use the general rule:

$$\Delta f(x, y, ...) = \sqrt{\left(\frac{\delta f}{\delta x}\right)^2 (\Delta x)^2 + \left(\frac{\delta f}{\delta y}\right)^2 (\Delta y)^2 + ...}$$

We have $g(l, T)$.

$$\begin{aligned}
\frac{\delta g}{\delta l} &= \frac{4\pi^2}{T^2} \\
\frac{\delta g}{\delta T} &= \frac{-2.4\pi^2 l}{T^3} \\
\text{sub these in} &: \\
\Delta g(l, T) &= \sqrt{\left(\frac{\delta g}{\delta l}\right)^2 (\Delta l)^2 + \left(\frac{\delta g}{\delta T}\right)^2 (\Delta T)^2} \\
\Delta g &= \sqrt{\left(\frac{4\pi^2}{T^2}\right)^2 (\Delta l)^2 + \left(\frac{8\pi^2 l}{T^3}\right)^2 (\Delta T)^2} \\
\text{divide through by g} &: \\
\frac{\Delta g}{g} &= \sqrt{\frac{T^4}{16\pi^4 l^2}\left(\frac{4\pi^2}{T^2}\right)^2 (\Delta l)^2 + \frac{T^4}{16\pi^4 l^2}\left(\frac{8\pi^2 l}{T^3}\right)^2 (\Delta T)^2} \\
\frac{\Delta g}{g} &= \sqrt{\left(\frac{\Delta l}{l}\right)^2 + 4\left(\frac{\Delta T}{T}\right)^2}
\end{aligned} \tag{2}$$

Note the factor of 4. But this isn't the same as if we were just combining the errors of three quantities that were multiplied or divided, l, T and T:

$$= \sqrt{\left(\frac{\Delta l}{l}\right)^2 + \left(\frac{\Delta T}{T}\right)^2 + \left(\frac{\Delta T}{T}\right)^2}$$

That is because T and T are the same thing - they are 100% correlated so we should add those errors linearly, not in quadrature. Remember in the last lecture we determined that the error on T^2 was $2T\Delta T$?
We can rewrite the error propagation equation (bringing 4 inside bracket) as

$$\frac{\Delta g}{g} = \sqrt{\left(\frac{\Delta l}{l}\right)^2 + \left(\frac{2\Delta T}{T}\right)^2}$$

$$\frac{\Delta g}{g} = \sqrt{\left(\frac{\Delta l}{l}\right)^2 + \left(\frac{2T\Delta T}{T^2}\right)^2}$$

Lets plug in the numbers. $g = 4\pi^2 \frac{0.651}{1.62^2}\left(1 \pm \sqrt{\left(\frac{0.001}{0.651}\right)^2 + 4\left(\frac{0.03}{1.62}\right)^2}\right) = 9.79 \pm 0.36$

4 Example 4 - Voltages

$V_{AB} = 0.104 \pm 0.001$ V, $V_{BC} + 1.492 \pm 0.001$ V, and $R = 4.69 \pm 0.01 k\Omega$
determine $R_x = \frac{V_{AB}.R}{V_{BC}}$ and ΔR_X.
Assume the voltages are measured with 2 different instruments.
That means the voltage errors are uncorrelated. Lets just use the formula straight off - all multiplicative so add relative errors in quadrature.

$$\Delta R_X = R_X \sqrt{\left(\frac{\Delta V_{AB}}{V_{AB}}\right)^2 + \left(\frac{\Delta V_{BC}}{V_{BC}}\right)^2 + \left(\frac{\Delta R}{R}\right)^2}$$
$$\Delta R_X = 0.327\sqrt{(\frac{0.001}{0.104})^2 + (\frac{0.001}{1.492})^2 + (\frac{0.01}{4.69})^2} \approx 0.003 k\Omega$$
$$R_x = 0.327 \pm 0.003\ k\Omega$$

5 The Importance of Realistic Errors

The goal is to find the most probable range corresponding to the measured value.

For example. An election opinion poll indicates that the conservatives have $51 \pm 0.2\%$ of the vote - Labour will get worried. However if the same poll said that the conservatives had $51 \pm 10\%$ of the vote they wouldn't pay much attention.

This shows the dangers of over-estimating you error. Its equally important not to underestimate it.

6 Three Sigma Test

Two quantities are said to be in agreement if they differ by less than 3σ.

6.1 Case 1 - comparing to theory

We assume the error on the theory is negligible

- Measurement 1: $g = 11.5 \pm 0.9 ms^{-2}$
- Measurement 2: $g = 10.3 \pm 0.1 ms^{-2}$
- Theory - expect $g = 9.8 ms^{-2}$

$\Delta(\text{Obs}_1 - \text{Exp}) = 1.7 \pm 0.9\text{ms}^{-2}$
$\frac{1.7}{0.9} = 1.9 \rightarrow$ Agreement within 2σ ie. probable.

$\Delta(\text{Obs}_2 - \text{Exp}) = 0.5 \pm 0.1\text{ms}^{-2}$
$\frac{0.5}{0.1} = 5 \rightarrow$ Disagree at 5σ ie. very unlikely.

6.2 Case 2 - compare two measurements

$\Delta(\text{Obs}_1 - \text{Obs}_2) = 1.2 \pm \sqrt{0.9^2 + 0.1^2} = 1.2 \pm 0.91\text{ms}^{-2}$
$\frac{1.2}{0.91} = 1.3 \rightarrow$ Agreement within 2σ ie. probable.
Note that the combined error is completely dominated by measurement 1. Now this works if the measurements are uncorrelated - ie. if the errors are just random statistical uncertainty. However, if some proportion of the allotted error is common to both measurements - ie. a systematic uncertainty, then this shouldn't be included in the difference as they cancel each other out.

- Measurement 1: $A = 10 \pm 1.0(\text{syst}) \pm 0.2(\text{stat})\text{m}$
- Measurement 2: $A = 11 \pm 1.0(\text{syst}) \pm 0.3(\text{stat})\text{m}$

To understand the accuracy of the individual measurements we would add all sources of uncertainty in quadrature - ie the statistic and systematic so we'd get:

- Measurement 1: $A = 10 \pm 1.02\text{m}$
- Measurement 2: $A = 11 \pm 1.04\text{m}$

However, if the systematic uncertainty is common to both measurements we should only consider the statistical components in the error on the difference:
$\Delta(\text{Obs}_2 - \text{Obs}_1) = 1 \pm \sqrt{0.2^2 + 0.3^2} = 1 \pm 0.36\text{m}$
So the difference is 2.8σ rather than 0.7σ if we used the total error.

7 Lab Reports

A good report should contain the following:

- Title
- Authors + affiliation (eg. QMUL)
- Abstract
  One short paragraph ∼100 words explaining what you did and the main result and conclusion
- Theory / Principles of operation Don't repeat yourself or other people - you can use references. Give derivations of formulae if they are relevant.

- Experimental Method
  This should be precise, defining ALL the quantities measured (it should provide enough information for another experimentalist to repeat the experiment).
  Think about a logical flow for the method which may differ from the chronological order you measured things in.
- Results including tables, graphs and error analysis
  Show all raw data. Estimate uncertainties. Graphs must have axis labels, legend, units, caption, title (see lecture 2)
  There shouldn't be pages and pages of calculation (reference the equations used in the theory section).
- Conclusion Repeat and quantify the main result. Is it what you expected? Does it agree with theory and other published measurements of the same quantity. What did you learn?

- References List all papers/books/websites used at the end. Plaigarism is not acceptable.

Report should be precise, concise, professional, not chatty. The aim is to communicate information: another experimentalist should be able to reproduce the results from your report. Presentation points:

- Use a word processor
- Use a spell checker - spelling mistakes aren't acceptable
- Check grammar
- Use an equation editor
- Use symbol font for $+ - \times \div \approx \pi\theta$ etc.
- *italicise* variable names
- Number the pages
- check superscripts and subscripts
- Quote values to appropriate number of significant figures
- Don't forget the units.

and other hints:

- Print out in plenty of time in case of printer disasters
- Backup on a USB
- Test printout to ensure fonts and formatting correctly reproduced (eg plots don't block text etc)