SCM Lecture 3 - teaching notes

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August 25, 2010

1 A single quantity

Suppose we measure a parameter, $T \pm \Delta T$ but we actually want to use T^2 in our calculations/plots. What is the error on T^2?

Well look what happens to a limiting value $f(T) = T^2$

$$f(T) + \Delta f(T) = (T + \Delta T)^2 = T^2 + (\Delta T)^2 + 2T\Delta T$$

Now if $\Delta T << T$ it is reasonable to ignore the $(\Delta T)^2$ term and we get

$$f(T) + \Delta f(T) \approx T^2 + 2T\Delta T$$

so

$$\Delta f(T) = 2T\Delta T$$

We quite often consider fractional uncertainties (sometimes called relative uncertainty). Suppose we measure $T = 100 \pm 10$

$$\frac{\Delta T}{T} = 0.1 = 10\%$$

So what is the relative error in $f(T)$ - well divide through by $f(T)$

$$\frac{\Delta f(T)}{f(T)} = \frac{2T\Delta T}{T^2} = 2\frac{\Delta T}{T}$$

ie. the fractional error in T^2 is twice the fractional error in T – 20%.

Note this expression corresponds to the derivative

$$f = T^2$$
$$\frac{df}{dT} = 2T$$
$$df = 2T.dT$$

compare that to what we had earlier

$$\Delta f = \left|\frac{df}{dT}\right| \Delta T$$

This rule works for all functions: multiplication:

$$f(L) = kL$$
$$\Delta f = \left|\frac{df}{dL}\right| \Delta L$$
$$\Delta f = k\Delta L$$

reciprocal:

$$f(L) = \frac{1}{L}$$
$$\Delta f = \left|\frac{df}{dL}\right| \Delta L$$
$$\Delta f = \frac{1}{L^2}\Delta L$$

See slide on definition of the derivative. Derivative of a function just gives the gradient - evaluate at given point for error associated with that point.

2 Combining errors - Correlated

What do we mean by correlated? If x increases, y increases. Example - our value depends on two times measured with the same clock and the uncertainty arises from inaccuracies in the clock - is it fast or slow. If it is fast, both times will be underestimated, if it is slow they will be overestimated. This is the worst case scenario - largest combined error.

2.1 General Rule

$$\Delta f(x, y) = \left|\frac{\delta f}{\delta x}\right| \Delta x + \left|\frac{\delta f}{\delta y}\right| \Delta y$$

2.2 Sums

$$f(x, y) = x \pm y$$
$$\Delta f = \Delta x \pm \Delta y$$

2.3 Products and Quotients

$$f(x, y) = x \times y$$

$$\Delta f = y\Delta x + x\Delta y$$

$$\frac{\Delta f}{f} = \frac{y\Delta x + x\Delta y}{xy} = \frac{\Delta x}{x} + \frac{\Delta y}{y}$$

3 Combining errors - Uncorrelated

The most probable error is smaller than the maximum error estimate we've been looking at so far. If x can vary between $x - \Delta x \rightarrow x + \Delta x$ and y can vary between $y - \Delta y \rightarrow y + \Delta y$ then it is unlikely that we will measure $x + \Delta x$ and $y + \Delta y$ at the same time. This is USUALLY the case. ie if x is above average it is equally likely that y is above or below average. For example - if we measure the length of a pendulum to be 2% longer than it should be, there is no reason why we should measure the period to be longer - its a completely different measurement. Therefore, so we don't overestimate the total uncertainty we combine the errors in quadrature.

$$\Delta f(x, y, ...) = \sqrt{\left(\frac{\delta f}{\delta x}\right)^2 (\Delta x)^2 + \left(\frac{\delta f}{\delta y}\right)^2 (\Delta y)^2 + ...}$$

compare this to the correlated case - we are adding the squares together and taking the square root of the total now. Think about a pythagoras triangle - the hypotenuse is shorter than the two contributing sides added together. Justification:

$$\begin{array}{rcl}
A &=& X.Y \\
\bar{A} &=& \frac{1}{N}\sum x_i.y_i \\
\bar{A} + \Delta A_i &=& (\bar{x} + \Delta x_i)(\bar{y} + \Delta y_i) \\
&=& \bar{x}.\bar{y} + \bar{x}\Delta y_i + \bar{y}\Delta x_i + \Delta x_i \Delta y_i \\
&\approx& \bar{x}.\bar{y} + \bar{x}\Delta y_i + \bar{y}\Delta x_i \\
\rightarrow \Delta A_i &=& \bar{x}\Delta y_i + \bar{y}\Delta x_i \\
\sigma_A &=& \sqrt{\frac{1}{N-1}\sum (A_i - \bar{A})^2} \\
\sigma_A^2 &=& \frac{1}{N-1}\sum (\Delta A_i)^2 \\
\sigma_A^2 &=& \frac{1}{N-1}\sum (\bar{x}\Delta y_i + \bar{y}\Delta x_i)^2 \\
&\approx& \frac{1}{N-1}\sum \left(\bar{x}^2 \Delta y_i^2 + \bar{y}^2 \Delta x_i^2\right) \\
&\approx& \frac{1}{N-1}\bar{x}^2 \sum \Delta y_i^2 + \bar{y}^2 \sum \Delta x_i^2 \\
\sigma_A^2 &=& \bar{x}^2 \sigma_y^2 + \bar{y}^2 \sigma_x^2 \\
\div A^2 \quad \left(\frac{\sigma_A}{A}\right)^2 &=& \left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2 \\
\frac{dA}{dx} = y \rightarrow \qquad \sigma_A^2 &=& \left(\frac{dA}{dx}\right)^2 \sigma_x^2 + \left(\frac{dA}{dy}\right)^2 \sigma_y^2
\end{array} \tag{1}$$

See slide with summary of basic cases. But can derive all of them from the basic formula. Adding in quadrature, the largest error contribution will always dominate. If one error is much smaller you may be able to neglect it.