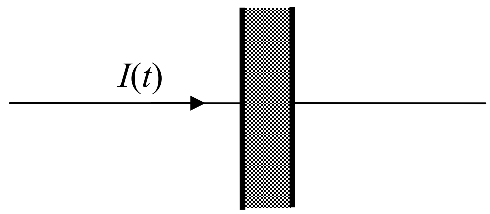
2B29 Electromagnetic Theory

4. Energy in Static Electromagnetic Fields

We will see later that when electromagnetic fields vary rapidly energy can be carried away as radiation. Before we can discuss that we need to establish how static or slowly varying electrical systems store energy.

4.1 Electrostatic Energy

It requires energy to establish either electric or magnetic fields. For the electrostatic case the argument is simple, based again on consideration of what is happening inside a parallel plate capacitor with gap *d*, area *A* and capacitance $C = \frac{\varepsilon_r \varepsilon_0 A}{d}$ (equ. (1.14)).



This may be a revision of material from 1B26. The volume of the dielectric inside the capacitor is $\tau = Ad$. If current $I(t)$ is flowing when the voltage across the capacitor is $V(t)$ then the work being done is $W = IV$. Integrating this from a situation with no charge stored on the capacitor, the stored energy is

$$S = \int_0^t IVdt = \int_0^t \frac{dQ}{dt}\frac{Q}{C}dt = \frac{Q^2}{2C} = \frac{QV}{2} \tag{4.1}$$

We discussed the size of **E** and **D** inside the capacitor in Section 1 just after equation (1.26) $\mathbf{D} = \varepsilon_r \varepsilon_0 \mathbf{E}$. The size of *D* depends only on the free charge *Q* that has come onto the plates from the external circuit, $D = Q/A$, and $E = V/d$. So the energy per unit volume inside the dielectric is

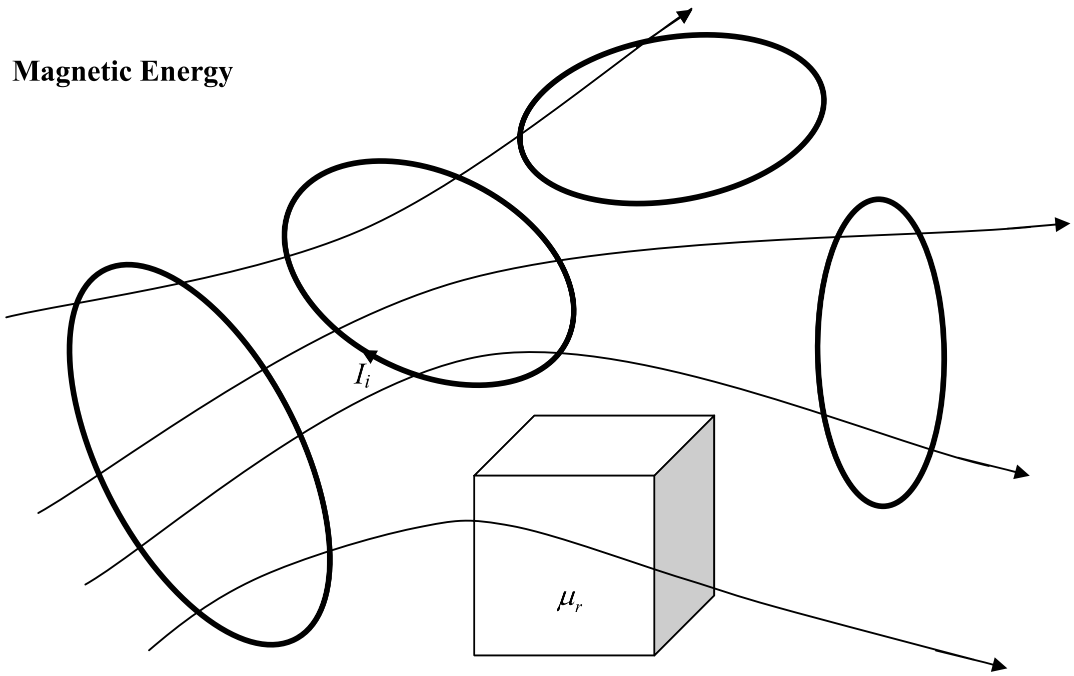
$$U_e \equiv \frac{S}{\tau} = \frac{QV}{2\tau} = \frac{1}{2}\frac{Q}{A}\frac{V}{d} = \tfrac{1}{2}DE. \tag{4.2}$$

This electrostatic energy density inside a linear isotropic dielectric material can be generalised to more complicated materials (not proven), as

$$U_e = \frac{dS}{d\tau} = \frac{1}{2}\mathbf{E.D}. \tag{4.3}$$

In vacuum $U_e = \varepsilon_r \varepsilon_0 E^2/2$.

4.2 Magnetic Energy



This argument looks very artificial but is actually quite general We consider an isolated region containing a number *n* of fixed electric circuits. The *i*th circuit has a battery which is supplying a slowly varying current $I_i(t)$ and the circuit is threaded by a flux $\Phi_i(t)$ due to the combined effect of all the circuits and of fixed pieces of magnetisable material in the region. For the purposes of argument we ignore ohmic losses in the circuits with steady currents.

The power coming from the *i*th battery to maintain the current at time *t* is

$$P_i(t) = I_i(t)V_i(t) = I_i(t)\frac{d\Phi_i}{dt} \text{ watts,} \tag{4.4}$$

where $\frac{d\Phi_i}{dt}$ is the rate of change of flux through the *i*th circuit which induces the voltage V_i that opposes the change (Faraday's law). Summing this over all the circuits in the region, the total work done by all of the batteries in time δt is

$$\delta W = \sum_{i=1}^{i=n} I_i \frac{d\Phi_i}{dt}\delta t\,, \tag{4.5}$$

This is equal to the increase in the stored magnetic energy in the field for the whole region.

Now assume that all the currents $I_i(t)$ are ramped up together at a steady rate over a time *T* to reach maximum values I_{m_i}, with $I_i(t) = \frac{I_{m_i}}{T}t$. If the only fluxes present are caused by the currents then they will also rise steadily to Φ_{m_i}, with $\Phi_i(t) = \frac{\Phi_{m_i}}{T}t$.

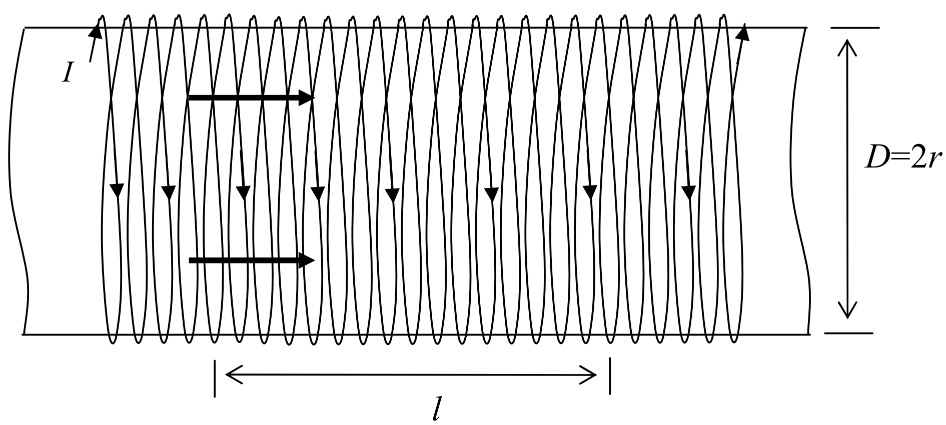
Then the energy stored in the whole system will be

$$W = \int_0^T \delta W = \sum_i \int_0^T I_i \frac{d\Phi_i}{dt} dt$$
$$= \sum_i \frac{I_{m_i} \Phi_{m_i}}{T^2} \int_0^T t dt$$

Integrating this we find that the energy stored in the magnetic field in the whole region is

$$W = \frac{1}{2} \sum_i I_{m_i} \Phi_{m_i} \tag{4.6}$$

We apply this to a special case, a long solenoid with *N* turns/m wound on a piece of magnetisable material (c.f. section 3.3). Each turn carrying current *I* can be regarded as a fixed circuit. Flux $\Phi = \pi r^2 B$ loops through every turn;
$B = B_{solenoid} + B_{magnet} = \mu_0 (NI + M)$ from equation (3.3). Since *H* depends only on free currents it has the same value in this electromagnet as it had in the solenoid in vacuum, $H = \frac{B_{solenoid}}{\mu_0} = NI$



There is no significant **B** field in the outside world, except in insignificant volumes very close to each end, so all the energy must be stored in the uniform field inside the solenoid magnet.

Equation (4.6) can now be rewritten to get the stored energy W_l in length *l* of this magnet, with $\sum_{lenght\ l} I_{m_i} = NlI$ and $\Phi_{m_i} = \Phi = \pi r^2 B$.

So
$$W_l = \frac{1}{2} \sum_{length\ l} I_{m_i} \Phi_{m_i} = \frac{1}{2} N \pi r^2 l I B. \tag{4.7}$$

But $\pi r^2 l$ is just the volume of length *l* of the magnet, and $NI = H$, so the energy density inside the magnetised material is

$$U_m = \frac{W_l}{\pi r^2 l} = \frac{1}{2} HB. \tag{4.8}$$

If this is true inside a solenoid there is no reason why it should not be true in all materials where **B**, **H** (and therefore **M**) are parallel to one another.

As for the electrostatic energy $U_e = \frac{1}{2}\mathbf{E.D}$, equation (4.3) , a more sophisticated argument is needed to treat the case where **B**, **H** and **M** are not parallel to one another. The most general result is then (see e.g. Jackson, "Classical Electromagnetism", Wiley)

$$U_m = \frac{1}{2}\mathbf{H.B} \tag{4.9}$$

4.3 Forces due to Magnetic Fields

These are literally what make the modern world work. We derive the basics.

Consider again the very general system of coils and magnetised stuff discussed in section 4.2 above. Using equation (4.9), the total magnetic energy within some volume τ will be

$$W_m = \int_\tau U_m d\tau = \frac{1}{2}\int_\tau \mathbf{H.B}d\tau . \tag{4.10}$$

If this is all contained in currents I_i and fluxes Φ_i then equation (4.6) gives

$$W_m = \frac{1}{2}\sum_i I_i\Phi_i . \tag{4.11}$$

If an element of the system moves the fluxes may change. Let a coil or a piece of magnetised material (assumed linear here and in any problems set) move by a distance δs, causing flux changes $\delta\Phi_i = \frac{\partial\Phi_i}{\partial s}\delta s$ in all circuits.

We assume that the batteries in the circuits supply power to maintain steady currents I_i against the transient voltages $\partial\Phi_i / \partial t$ which will be induced by the flux changes. But the motion of an element may also have done external work, and this must also be provided by the batteries. Alternatively, the motion of an element may do work on the system and the batteries may receive energy from this (think of charging a battery by pedaling a bicycle).

The power coming from the batteries is $\sum_i I_i \frac{\partial\Phi_i}{\partial t}$ so the energy from the batteries in time δt while the movement δs happens is $\delta W_b = \sum_i I_i \int_0^{\delta t} \frac{\partial\Phi_i}{\partial t}dt = \sum_i I_i\delta\Phi_i$ (since the integrand is perfect derivative).
So

$$\delta W_b = \sum_i I_i\delta\Phi_i = \delta W_m + \delta W_f \tag{4.12}$$

where we have divided the energy supplied by the battery into the part δW_m which goes into the stored energy of the magnetic field, and the part δW_f which goes into mechanical work by force F_s acting on the element moving δs in the direction of the force;

$$\delta W_f = F_s \delta s \tag{4.13}$$

But from (4.11)

$$\delta W_m = \frac{1}{2}\sum_i I_i \delta\Phi_i \tag{4.14}$$

since the currents are maintained steady and only the fluxes change. So, from (4.12) and (4.14)

$$\delta W_f = \delta W_b - \delta W_m = \frac{1}{2}\sum_i I_i \delta\Phi_i = \delta W_m . \tag{4.15}$$

The work done by the force is exactly half of the work done by the batteries; the rest of the energy goes into increasing the stored magnetic field. [The middleman takes ½, as always!]

So the force on an element which is free to move in one dimension is just

$$F_s = \frac{\partial W_m}{\partial s} . \tag{4.16}$$

Or, if the element is allowed to move in three orthogonal dimensions then we can treat each separately to get $F_x = \dfrac{\partial W_m}{\partial x}$ etc., so

$$\mathbf{F} = \nabla W_m \tag{4.17}$$

When the element moves $\delta\mathbf{s}$ the work done is $\mathbf{F}.\delta\mathbf{s}$, according to whether the stored energy is increasing or decreasing.

(You may be set problems in which a gap s is opened in the yoke of an electromagnet. The stored magnetic energy can be calculated as a function of the gap spacing and this can be differentiated to get the force exerted by the magnet.)