2B29 Electromagnetic Theory

1. Introduction; Reminder of 1B26

1.1 Two Fundamental Fields

The two fundamental fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ are the only ones that matter in a vacuum. What are their names?

What are their units?

We can find the dimensions of those units by going back to a fundamental defining equation. For instance, the Lorentz force equation

$$\mathbf{F}(\mathbf{r}) = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1.1}$$

F is the force on a moving charge with velocity **v** at position **r**. So, for example, the dimensions of **B** in terms of fundamental quantities [mass] = M, [length] = L, [time] = T, [charge] = Q will be given by

$$[\mathbf{B}] = [\mathbf{F}]/[q][\mathbf{v}] \tag{1.2}$$

so

$$[\mathbf{B}] = (MLT^{-2})/QLT^{-1} = MQ^{-1}T^{-1} \tag{1.3}$$

1.2 Electrostatics

Two charges q_1 and q_2, at rest. The force between them is given by Coulomb's law

$$\mathbf{F} = \frac{q_1 q_2 \hat{\mathbf{r}}_{12}}{4\pi\varepsilon_0 r_{12}^2} \tag{1.4}$$

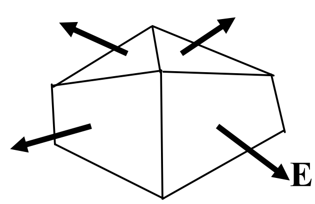
and for a static system with a number of charges, the field due to all the other charges at the charge q_j, position $\mathbf{r}_j$, is

$$\mathbf{E}(\mathbf{r}_j) = \frac{\mathbf{F}(\mathbf{r}_j)}{q_j} = \frac{1}{4\pi\varepsilon_0} \sum_{i \neq j} \frac{q_i}{r_{ji}^2} \hat{\mathbf{r}}_{ji} \tag{1.5}$$

where $\mathbf{r}_{ji} = \mathbf{r}_i - \mathbf{r}_j$. We call ε_0 the permittivity of free space.

For charges enclosed within a surface S we have Gauss' law

$$\oint \mathbf{E}.d\mathbf{S} = \frac{1}{\varepsilon_0} \sum_i q_i \tag{1.6}$$



The outward flux of **E** through S is given by the sum of all the charges within S. The elements $d\mathbf{S}$ in the integral are vectors with size equal to a small part of the area S, along the local outward normal $\hat{\mathbf{n}}$ to the surface.

If the charges are present as a continuous volume density $\rho(\mathbf{r})$ {what are the units of ρ?)
then we can turn the r.h.s. of (1.6) into an integral over the volume V inside S to get

$$\oint_S \mathbf{E}.d\mathbf{S} = \frac{1}{\varepsilon_0}\int_V \rho d\tau \qquad (1.7)$$

where $d\tau$ is an element of volume. Now we can use Gauss' divergence theorem from our Mathematical Tools, for any vector field **A**:

$$\int_V \nabla\cdot\mathbf{A}\,\mathrm{d}\tau = \oint_S \mathbf{A}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \oint_S \mathbf{A}\cdot\mathrm{d}\mathbf{S} \qquad (1.8)$$

so if we take $\mathbf{E}(\mathbf{r}) = \mathbf{A}(\mathbf{r})$ then $\int_V \nabla\cdot\mathrm{E}\,\mathrm{d}\tau = \frac{1}{\varepsilon_0}\int_V \rho d\tau$ (1.9)

But (1.9) must be true for any arbitrary volume *V,* including a vanishingly small volume around a point **r,** so we can conclude that

$$\nabla\cdot\mathrm{E} = \frac{\rho}{\varepsilon_0} \qquad (1.10)$$

that is, we've gone from (1.7), a statement about the flux of **E** out of the walls of a finite volume *V,* where $\rho(\mathbf{r})$ can vary from place to place, to an equation relating the charge density at a point **r** to the divergence of **E**(**r**) at that point. When the divergence of a vector field is related like this to the value of a scalar field $\rho(\mathbf{r})$ at the same point, we say that the scalar $\rho(\mathbf{r})$ is the source of **E**(**r**). (1.10) is not yet quite what we will remember as one of Maxwell's equations.

For static **E** fields we can define an electrostatic potential

$$\phi(\mathbf{r}) = \int_{\mathbf{o}}^{\mathbf{r}} \mathbf{E}.d\mathbf{l} \qquad (1.11)$$

where **o** is an arbitrary origin and the line integral *d***l** can be along any arbitrary path. This corresponds to the work done against the field in moving unit +ve charge from **o** to **r.**

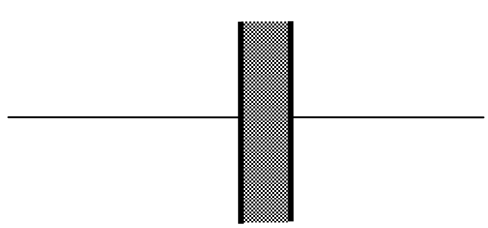
The differential version of this is $\mathbf{E} = -\nabla\phi$ (1.12)

where the minus sign represents loss of potential energy if the test charge moves in the direction of **E.** The potential difference $\phi(\mathbf{r}_1) - \phi(\mathbf{r}_2)$ between points $\mathbf{r}_1$ and $\mathbf{r}_2$ is measured in Volts.

1.3 Electric Fields in Materials

That is all fine in a vacuum, but what happens when we do electrostatics inside a dielectric medium? (We'll assume a linear medium. It gets much more complicated in the nonlinear case – but that is more important for magnetism, later in these lectures.)

Think back to the discussion of capacitors in 1B26. When we put a slab of a dielectric inside a parallel plate capacitor



then for the same voltage V across the plates we can store more charge Q. (N.B. this subject needs so many symbols that we can never completely avoid re-using them, like V for volume and V for volts.) That means that the capacitance $C = Q/V$ goes up when the dielectric is inserted. We call the dimensionless factor by which it increases the dielectric constant κ. The formula relating capacitance to gap spacing d and plate area A, in vacuum, is

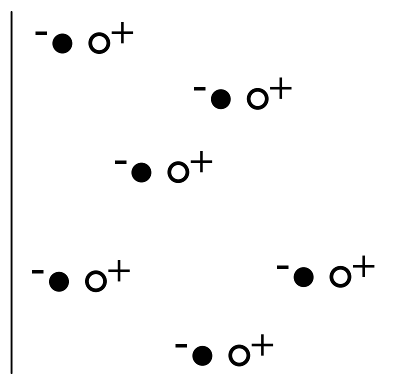
$$C = \frac{\varepsilon_0 A}{d} \tag{1.13}$$

So when we put the material in it becomes

$$C = \frac{\kappa\varepsilon_0 A}{d} \tag{1.14}$$

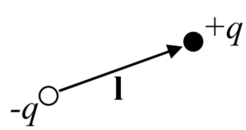
(If the material were nonlinear, how would that show up in (1.14)?)

What has happened in the material? For linear materials it is a good model to imagine the dielectric being composed of a very large number of tiny electric dipoles, which might be identified as individual electrons, which can move a little bit, each tied to its positive ion which stays at a fixed place in the lattice of the material.



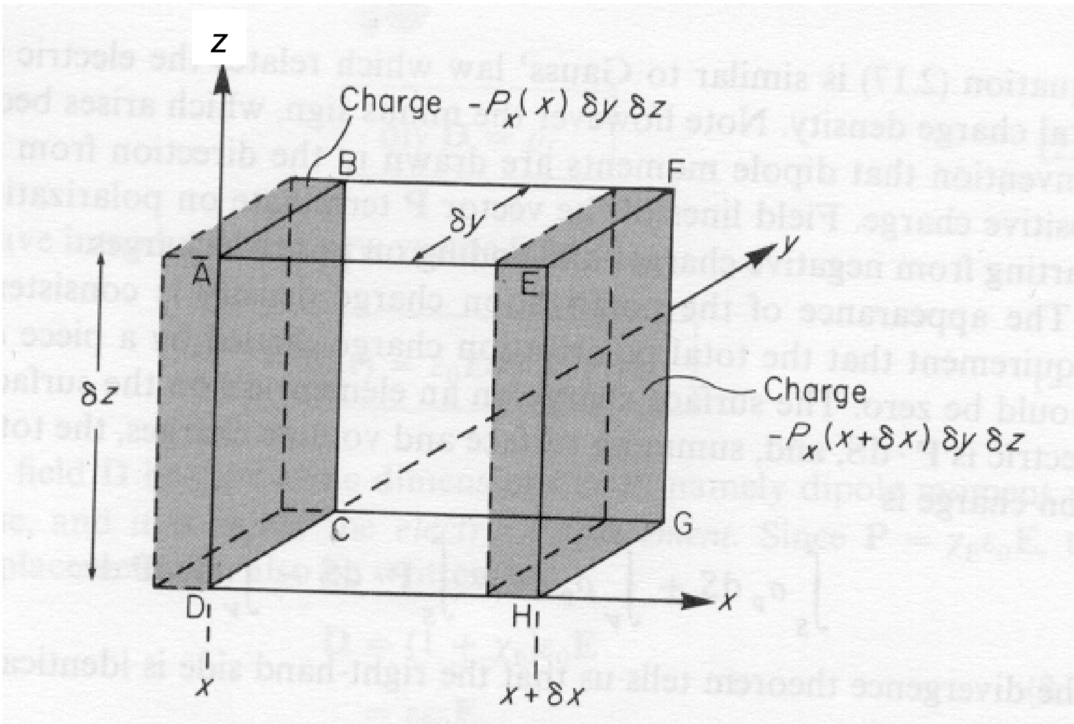
Dielectric Slab in uniform E field in vacuum.
(dipole sizes much exaggerated)

The electrical effects of a dipole can be described in terms of the dipole moment $\mathbf{m} = q\mathbf{l}$, where there are charges $+q$ and $-q$ at the two ends, with vector $\mathbf{l}$ from –ve to +ve.



In a linear medium the effect of increasing **E** will be to increase the displacement of each electron from its ion, hence strengthening the dipole moments. When we have lots of dipoles crammed into a small region their effects combine to give a dipole moment per unit volume which can vary from place to place. This is the polarisation **P**(**r**), another vector field. (What are its units?) If the **E** field is uniform, as in a parallel plate capacitor, then the dipoles inside the material are uniformly packed and **P** is uniform too. At the surfaces **P** falls off rapidly to zero. .

Following Grant and Phillips (Duffin is equivalent, but not so clear) we can use a simple Cartesian coordinate system to discuss the increase of **P** as **E** increases.



Assume a positive x component $P_x(x)$ of the polarisation at the plane ABCD. As **E** increases from zero electrons move out of the volume leaving their positive ions behind. If there are N dipoles per unit volume, each with charge $\pm q$ and displacement a in the x direction, then $P = Nqa$. The charge which has crossed surface ABCD as this polarization was produced must have been $Nqa\delta y\delta z = -P_x(x)\delta y\delta z$, i.e. all electrons which started within distance $+a$ from the surface. If the **E** field is varying from place to place, then $P_x(x+\delta x)$ on the plane EFGH can have a different value from $P_x(x)$. Charge $-P_x(x+\delta x)\delta y\delta z$ enters the cube at this face. So if the variations of **P** are smooth on a macroscopic scale, and if δx, δy and δz are very small, but much bigger than atomic dimensions, the net charge entering the cube across ABCD and EFGH is

$$-\{P_x(x+\delta x)\delta y\delta z - P_x(x)\delta y\delta z\} = -\frac{\partial P_x}{\partial x}\delta x\delta y\delta z \tag{1.15}$$

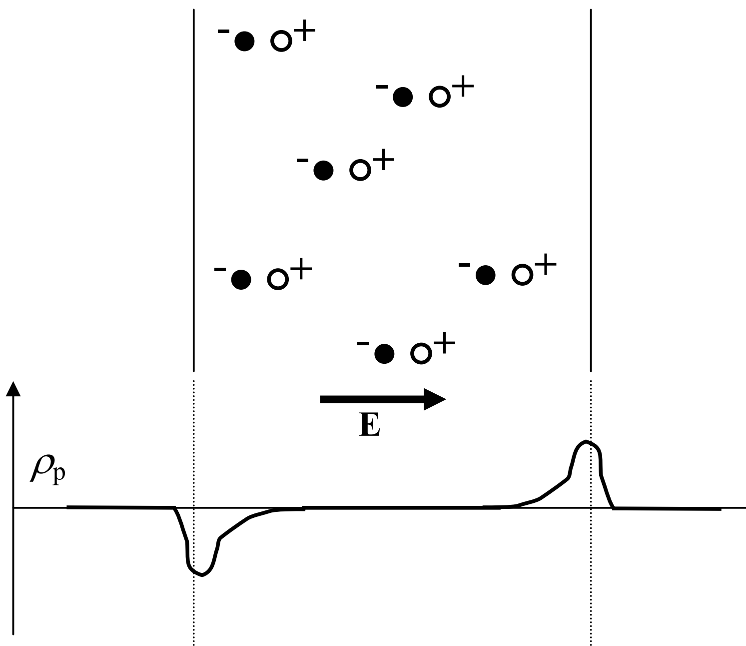
There can be similar contributions across the faces ABFE and DCGH from variations of P_z in the z direction, and across AEHD and BFGC for variations in the y direction. So the total charge acquired by the cube due to polarisation is

$$\left\{-\frac{\partial P_x}{\partial x} - \frac{\partial P_y}{\partial y} - \frac{\partial P_z}{\partial z}\right\}\delta x\delta y\delta z\,. \tag{1.16}$$

If we divide this by the volume element $\delta x\delta y\delta z$ then there is an effective "polarisation charge" density in the material

$$\rho_p = -\frac{\partial P_x}{\partial x} - \frac{\partial P_y}{\partial y} - \frac{\partial P_z}{\partial z} = -\nabla.\mathbf{P} \tag{1.17}$$

For the slab of dielectric in a parallel plate capacitor, deep inside away from the surfaces, **E** is uniform so **P** is uniform. In any tiny cube, as many electrons have moved in from one face as have moved out from the other, so the polarisation charge density $\rho_p = -\nabla \mathbf{P} = 0$ there. The sketched graph under the picture of the slab below shows the kind of thing that happens very close to the surfaces. On the left **-**ve charges can move closer to the surface, but there is nowhere else for the –ve charges which were already on the surface to go, so the negative charge piles up there producing a locally finite region of negative polarisation charge density. Something similar happens on the right where electrons move out, leaving fixed +ve charge behind them. In the thin surface layers **P** drops rapidly to zero, so it has nonzero divergence and there is a finite value of ρ_p. Since the surface layers are so thin we often, for convenience, regard the polarisation charge in this case as having only a surface density ($[\rho_{surf}] = QL^{-2}$) at the faces of the slab. Going back to the argument before equation (1.15), the amount of charge that accumulates at a surface ABCD is $Nqa\delta y\delta z = -P_x(x)\delta y\delta z$, so the charge density on a surface $\perp$ to the x-direction is just $Nqa = P_x$. For a general surface with an outward normal $\hat{\mathbf{r}}$ it is easy to show that the surface charge density is $\mathbf{P}.\hat{\mathbf{r}}$.



Length scale exaggerated near surface of slab.

1.4 The Electric Displacement D

We have now talked about two different kinds of charge density. In (1.10) the charge density $\rho(r) = \varepsilon_0 \nabla \cdot \mathrm{E}$ is *a combination of* the polarisation charge density ρ_p from (1.17), generated by charges tethered to their positions in the material, *plus* the free charge density ρ_f due to charges that can move around, like charges in a vacuum or in a metal. So we can write

$$\nabla \cdot \mathrm{E} = \frac{\rho_p + \rho_f}{\varepsilon_0} \tag{1.18}$$

and, using (1.17)

$$\rho_f = \varepsilon_0 \nabla \cdot \mathrm{E} - \rho_p = \nabla \cdot (\varepsilon_0 \mathrm{E} + \mathbf{P}) \tag{1.19}$$

Equation (1.19) says that the free charge is equal to the divergence of a quantity $(\varepsilon_0 \mathrm{E} + \mathbf{P})$. It then becomes very convenient to define a new *secondary field*, the electric displacement

$$\mathbf{D}(\mathbf{r}) \equiv (\varepsilon_0 \mathrm{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})) \tag{1.20}$$

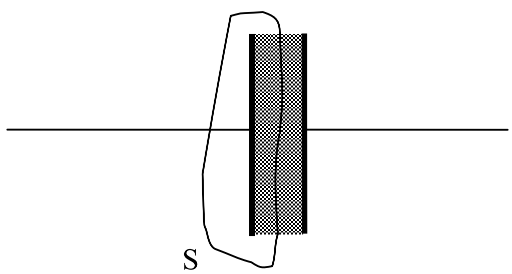
so that

$$\nabla \cdot \mathbf{D} = \rho_f \tag{1.21}$$

We will use this, rather than (1.10), as one of the four Maxwell equations. It will allow us to treat electromagnetic waves in linear media as easily as waves in vacuum.

Intuitively, the thing to remember about the displacement **D** is its direct relationship to *free* charge. The electric field strength **E** is generated by the total charge, including free and polarisation charge, as in (1.18).

Tidying up definitions, we go back to the parallel plate capacitor. When we fill it with material with dielectric constant κ, holding a fixed stored charge Q on the plates, the voltage V across the plates goes down by a factor $1/\kappa$ since C has increased by κ. Thus E (=$|\mathbf{E}|$) inside the material has decreased by $1/\kappa$ because the polarisation charges on the two surfaces now screen out some of the free charge Q. From inside the dielectric it seems that the effective charge at each surface is Q-Q_P, where $Q_P = PA$ is the amount of charge that concentrates at each face of the dielectric when polarisation P (=$|\mathbf{P}|$) is produced (c.f. argument before (1.15)).



We can calculate the flux of **E** through a Gaussian surface S like a sealed bag completely enclosing one plate but cutting through the dielectric. Assuming **E** is only significant inside the capacitor, and is uniform and perpendicular to the plates, then from (1.9)

$$\oint_S \mathbf{E}.d\mathbf{S} = EA = \frac{Q - Q_P}{\varepsilon_0} = \frac{Q - PA}{\varepsilon_0}\,. \tag{1.22}$$

When there is no dielectric $E_{vac}A = Q/\varepsilon_0$. But $E_{vac} = \kappa E$

so

$$EA = \frac{\kappa\varepsilon_0 EA - PA}{\varepsilon_0} \tag{1.23}$$

and

$$P = \varepsilon_0(\kappa - 1)E\,. \tag{1.24}$$

We define the electrical susceptibility of the material to be $\chi_E \equiv (\kappa - 1)$. More generally, for isotropic linear materials, we can write

$$\mathbf{P} = \chi_E \varepsilon_0 \mathbf{E} = \varepsilon_0(\kappa - 1)\mathbf{E}\,. \tag{1.25}$$

(What are the dimensions of χ_E? Why mention "isotropic and linear"?)

Notice that, from (1.20) and (1.25)

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \varepsilon_0 (\kappa - 1)\mathbf{E} = \kappa \varepsilon_0 \mathbf{E}$$

where κ is a dimensionless multiplier of the vacuum permittivity ε_0. That leads us to the notation which we will use from now on. We define the relative permittivity $\varepsilon_r \equiv \kappa$, and the permittivity of a medium $\varepsilon = \varepsilon_r \varepsilon_0$, so that

$$\mathbf{D} = \varepsilon_r \varepsilon_0 \mathbf{E} = \varepsilon \mathbf{E}. \tag{1.26}$$

Now we can also write $\chi_E \equiv (\varepsilon_r - 1)$.

[**Look out**. Some books use ε to mean our ε_r]

If we are looking for an intuitive feeling for **D**, we see that in the case above where we are holding the charge Q constant on the plates of the capacitor:

a) when there is a vacuum between the plates

$$D = \varepsilon_0 E_{vac} = Q / A.$$

b) when the space is filled with dielectric

$$D = \varepsilon_r \varepsilon_0 E = \varepsilon_r \varepsilon_0 E_{vac} / \varepsilon_r = \varepsilon_0 E_{vac} = Q / A, \text{ again.}$$

So the **D** field is governed only by the free charge Q on the plates and is unchanged when dielectric is inserted, whereas the **E** field (and hence the voltage across the capacitor) is reduced by the presence of dielectric.

[You will be set a tutorial question to show how easily $D = Q / A$ comes out from the Gaussian integral used at equation (1.22)]

1.5 Magnetostatics

This could be as important and intricate as electrostatics – if we lived in a universe containing lots of magnetic monopoles. But we haven't found any yet. Until we do we can assume that their density ρ_M is zero so, by analogy with (1.7) and (1.10) we get

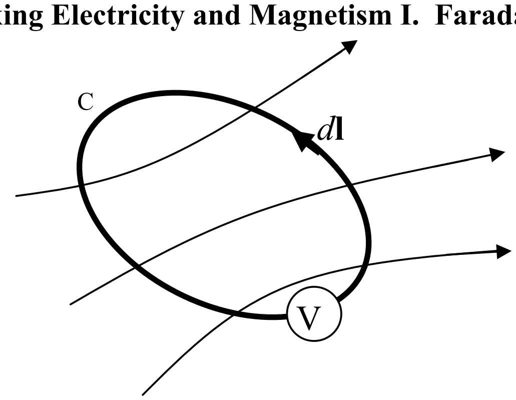
$$\int_S \mathbf{B}.d\mathbf{S} = 0 \tag{1.27}$$

and

$$\nabla.\mathbf{B} = 0. \tag{1.28}$$

This is another Maxwell equation.

1.6 Mixing Electricity and Magnetism I. Faraday's Law of Induction



A conducting circuit C may be intersected by a **B** field which gives a flux

$$\Phi_C = \int_{S_C} \mathbf{B}.d\mathbf{S} \tag{1.29}$$

through the surface S_C contained by C. Then the emf induced around the circuit is

$$V = -\frac{d\Phi_C}{dt} = \oint_C \mathbf{E}.d\mathbf{l}\,, \tag{1.30}$$

integrating over all line elements $d\mathbf{l}$ in C. (Note that this non-zero value for a circuit integral of $\mathbf{E}$ appears to suggest that the potential in (1.11) is not unique; we could increase its value arbitrarily by doing a series of loop integrals. But section 1.2 was concerned only with static electric fields without any changing magnetic fluxes. What do we call a field where all loop integrals are zero?)

Stokes' theorem says (see Tools) $\int_{S_C} (\nabla\times\mathbf{A}).d\mathbf{S} = \oint_C \mathbf{A}.d\mathbf{l}$. So from (1.29) and (1.30) with $\mathbf{A}\equiv\mathbf{E}$ we get $\oint_C \mathbf{E}.d\mathbf{l} = -\frac{d\Phi_C}{dt} = -\frac{d}{dt}\int_{S_C}\mathbf{B}.d\mathbf{S} = \int_{S_c}(\nabla\times\mathbf{E}).d\mathbf{S}$

and

$$-\int_{S_C}\frac{d\mathbf{B}}{dt}.d\mathbf{S} = \int_{S_c}(\nabla\times\mathbf{E}).d\mathbf{S} \tag{1.31}$$

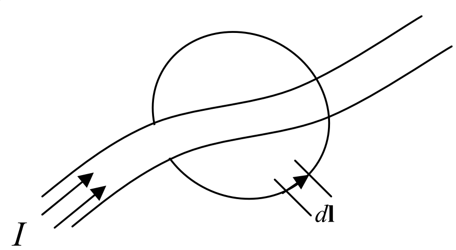
We can use this in the same way that we did (1.9). The circuit C and its contained surface S_C can be shrunk to an arbitrarily small size about any point $\mathbf{r}$, within which region the integrands on either side of (1.31) will have negligible variation, so they must be equal; that is

$$\nabla\times\mathbf{E} = -\frac{d\mathbf{B}}{dt} \tag{1.32}$$

This is another of Maxwell's equations.

1.7 Mixing Electricity and Magnetism II: Ampere's Law.

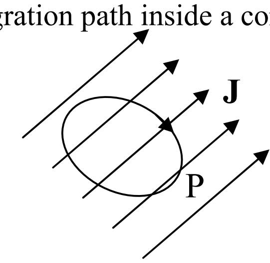
We will get our fourth Maxwell equation from Ampere's law, but we have some more theory to develop before that. For now, let's remind ourselves of the 1B26 version.



For steady current I through a conductor surrounded by vacuum we have

$$\oint_P \mathbf{B}.d\mathbf{l} = \mu_0 I\ . \tag{1.33}$$

where the closed integration path P completely encloses the conductor. Or for an integration path inside a conducting region with current density $\mathbf{J}(\mathbf{r})$



$$\oint_P \mathbf{B}.d\mathbf{l} = \mu_0\int_{S_P}\mathbf{J}.d\mathbf{S} \tag{1.34}$$

both (1.33) and (1.34) say that the integral of $\mathbf{B}/\mu_0$ around a closed loop is equal to the current flowing through the area of the loop.

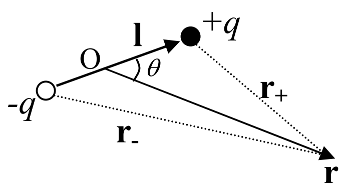
But Stokes' theorem says $$\oint_P \mathbf{B}.d\mathbf{l} = \int_{S_P} (\nabla\times\mathbf{B}).d\mathbf{S} \tag{1.35}$$

so $$\nabla\times\mathbf{B} = \mu_0\mathbf{J}\,. \tag{1.36}$$

by the same kind of argument we used to derive (1.32). This is an incomplete form of the last Maxwell equation.

1.8 The Field of a Static Electric Dipole

(You should understand this argument, but the whole detailed derivation below will not be asked for in an examination)



Origin at O in centre of dipole, length **l**

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{\hat{\mathbf{r}}_+}{r_+^2} - \frac{\hat{\mathbf{r}}_-}{r_-^2}\right]. \tag{1.37}$$

The squares and vectors make this tricky to evaluate. It is easier to use the electrostatic potential

$$\phi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_+} - \frac{1}{r_-}\right]. \tag{1.38}$$

Can get $\mathbf{E}(\mathbf{r})$ via (1.12), $\mathbf{E} = -\nabla\phi$, with $\mathbf{r}_\pm = \mathbf{r}\mp\frac{\mathbf{l}}{2}$ and $\frac{1}{r_\pm} = (|\,\mathbf{r}_\pm\,|^2)^{-1/2}$.

But $|\,\mathbf{r}_\pm\,|^2 = |\,\mathbf{r}\,|^2 \mp \mathbf{r}.\mathbf{l} + \frac{|\,\mathbf{l}\,|^2}{4} = r^2 \mp rl\cos\theta + \frac{l^2}{4} = r^2\left(1\mp\frac{l}{r}\cos\theta + \frac{l^2}{4r^2}\right)$.

So we can do a binomial expansion to get

$$\frac{1}{r_\pm} = \frac{1}{r}\left(1\mp\frac{l}{r}\cos\theta+\frac{l^2}{4r^2}\right)^{-1/2} = \frac{1}{r}\left(1\pm\frac{l}{2r}\cos\theta + X\frac{l^2}{r^2} + Y\frac{l^3}{r^3} + ...\right).$$

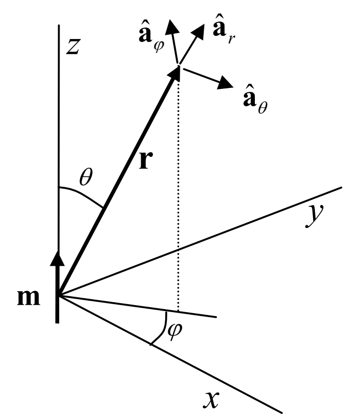
Neglecting all higher powers of l/r, we get $\left[\frac{1}{r_+} - \frac{1}{r_-}\right] \simeq \frac{l}{r^2}\cos\theta$, if $r >> l$. So from

(1.38) $$\phi(\mathbf{r}) \simeq \frac{q}{4\pi\varepsilon_0}\left(\frac{l}{r^2}\cos\theta\right) = \frac{\mathbf{m}.\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2}\,, \tag{1.39}$$

with $\mathbf{m} = q\mathbf{l}$, as before, and $\mathbf{m}.\mathbf{r} = mr\cos\theta$.

If we point such a dipole along the z axis of a spherical polar coordinate system we can explicitly calculate the div of the potential ϕ.

Unit vectors $\hat{\mathbf{a}}_r, \hat{\mathbf{a}}_\theta, \hat{\mathbf{a}}_\varphi$ are in the direction of motion of the point of $\mathbf{r} \equiv (r,\theta,\varphi)$ when each variable is increased with the other two held constant.



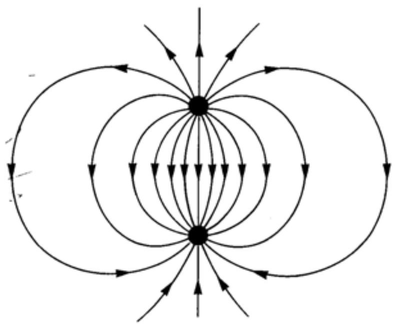
From Tools $$\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r}) = -\left(\hat{\mathbf{a}}_r \frac{\partial\phi}{\partial r} + \hat{\mathbf{a}}_\theta \frac{1}{r}\frac{\partial\phi}{\partial\theta} + \hat{\mathbf{a}}_\varphi \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\right). \quad (1.40)$$

Can also write $$\mathbf{E}(\mathbf{r}) = \hat{\mathbf{a}}_r E_r + \hat{\mathbf{a}}_\theta E_\theta + \hat{\mathbf{a}}_\varphi E_\varphi .$$

From(1.39) $\phi(\mathbf{r}) \simeq \dfrac{m\cos\theta}{4\pi\varepsilon_0 r^2}$ so $\dfrac{\partial\phi}{\partial\varphi} = 0$. Hence $E_\phi = 0$; (1.41)

$$\frac{\partial\phi}{\partial r} \simeq -\frac{2m\cos\theta}{4\pi\varepsilon_0 r^3}. \text{ Hence } E_r \simeq \frac{m\cos\theta}{2\pi\varepsilon_0 r^3} \quad (1.42)$$

$$\frac{\partial\phi}{\partial\theta} \simeq -\frac{m\sin\theta}{4\pi\varepsilon_0 r^2}. \quad \text{Hence } E_\theta \simeq \frac{m\sin\theta}{4\pi\varepsilon_0 r^3}. \quad (1.43)$$



1.9 Screening; falloff of field strength with distance

From (1.42) and (1.43) we see that the **E** field due to a dipole of given strength *m* falls off like $1/r^3$ while, from (1.5), the **E** field due to a single charge of given strength *q* falls off like $1/r^2$. At a large distance we say that the two opposite charges in the dipole "screen" one another, giving a joint effect which falls off faster than the effect of either of the charges on its own. Another way of looking at it is to say that two close charges look more and more like a neutral object as we go further away from them.