2B29 Electromagnetic Theory

14. Radio Wave Emission from a Hertzian Dipole

(Semi-quantitative. Need vector potential **A** for proper derivation)

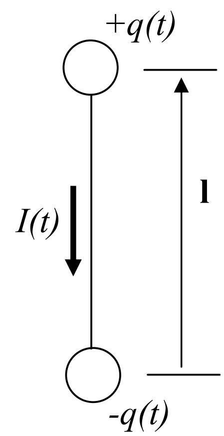
14.1 Near Field Picture

The Hertzian dipole is slightly idealised. It is formed from two identical balls with finite capacitance, connected by a thin straight conductor of length l and negligible capacitance. We therefore assume that when there is charge $+q$ on one ball there will be $-q$ on the other ball and no significant charge on the connector. At any instant

$$I(t) = -\frac{\partial q}{\partial t} \qquad (14.1)$$

The electric dipole moment at any instant is

$$\mathbf{m}(t) = q(t)\mathbf{l} \qquad (14.2)$$



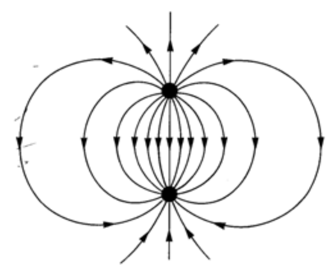
Close to the dipole the ***E*** field shape at any instant is given by equations (1.41), (1.42) and (1.43) from the introductory section of these lectures.

If we drive the dipole with an AC signal carried to it by a coaxial transmission line which connects to the centre of the straight conductor then we can impose from outside that

$$I = I_0 \sin \omega t \qquad (14.3)$$

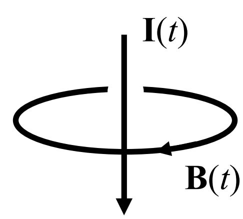
which can be integrated to give

$$q(t) = q_0 \cos \omega t \qquad (14.4)$$



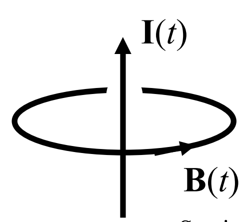
a) At $\omega t = 0$ there will be maximum +ve charge on the upper ball and –ve on the lower ball, giving the familiar electrostatic **E** field pattern, with zero current instantaneous flowing.

b) At $\omega t = \frac{\pi}{2}$, i.e. ¼ cycle later, the charges on the two balls will be zero and I will be maximal. Close to the dipole there is then no electrostatic field but there is a magnetic **B** field around the current.



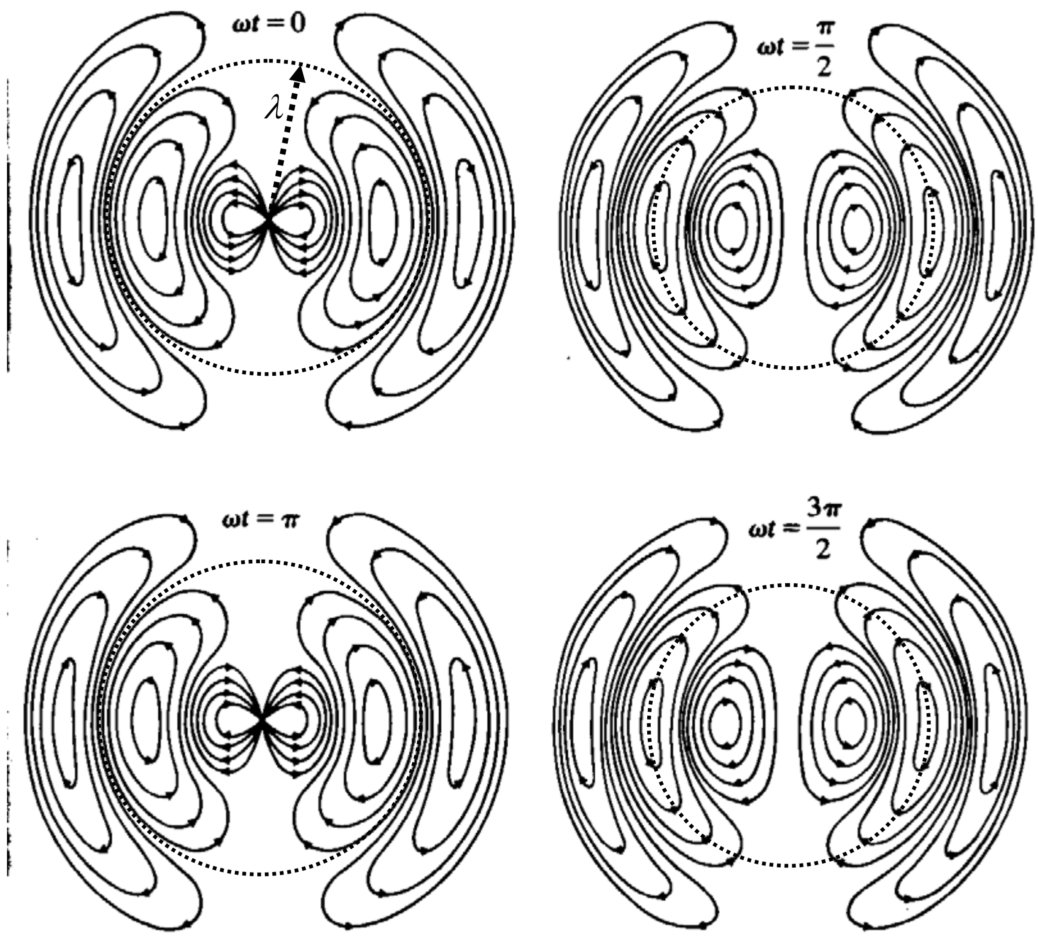
c) At $\omega t = \pi$ the charges on the balls will again be maximal but with the opposite sign, and the electrostatic **E** field pattern will be as at stage a) above, with arrows reversed. The current will again be zero.

d) At $\omega t = \frac{3\pi}{2}$ the charges will be zero. **I** and **B** will have reversed from stage b).



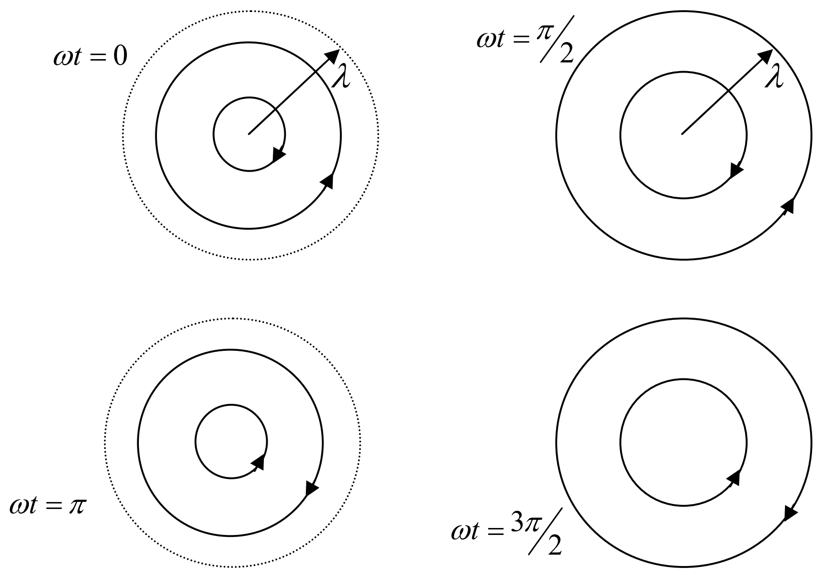
14.2 Intermediate Field Picture

The information that the dipole moment has changed with time propagates outwards at the speed of light c. The diagram below shows how the electrostatic field **E** varies with distance and time at distances comparable with the free space wavelength corresponding to the frequency ω. The dotted circle is at a distance $r = c\tau = \dfrac{2\pi c}{\omega} = \lambda$, the free space wavelength at that frequency. The arrows representing the direction of **E** reverse every half wavelength (in space) or every half period (in time).



[Note that the lines of force no longer go back to the charges which generated them. Instead they form closed "kidney shaped" loops in space which travel outwards at the speed of light c.]

In the equatorial plane of the dipole the **B** field which results from the current **I**(t) also reverses every half cycle as the rings of lines of **B** move outward at speed c. In the sketches below the inermost ring at $\omega t = 0$ moves outward from sketch to sketch to reach $r = \lambda$ at $\omega t = {}^{3\pi}\!/_{2}$. A new ring with reversed polarity is the innermost at $\omega t = \pi$.



To get the state of the **E** and **B** fields at a distance **r** from the centre of the dipole at time t we need to know the state of the dipole itself at a time r/c earlier; that is at the

retarded time
$$t' = \left(t - \frac{r}{c} \right). \tag{14.5}$$

At that time the electric dipole moment was

$$\mathbf{m}(t') = q(t')\mathbf{l} = \mathbf{l}q_0 \cos \omega t' \tag{14.6}$$

We put this into the expression for the electrostatic potential ϕ that we used in section 1 when we discussed the electric dipole

(1.39)
$$\phi(\mathbf{r}) \simeq \frac{q}{4\pi\varepsilon_0}\left(\frac{l}{r^2}\cos\theta \right) = \frac{\mathbf{m}.\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2},$$

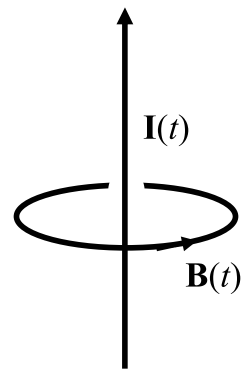
so we get
$$\phi(\mathbf{r},t) \simeq \frac{q_0 l \cos\theta}{4\pi\varepsilon_0 r^2}\cos\omega t' = \frac{q_0 l \cos\theta}{4\pi\varepsilon_0 r^2}\cos\left(\omega t - \frac{\omega r}{c} \right) \tag{14.7}$$

The cos expression on the right hand side is just the real version of the propagation factor for a wave moving towards increased r, with $k = {}^{\omega}\!/_{c} = 2\pi/\lambda$. [Remember; it doesn't matter if we write $(kr - \omega t)$ or $(\omega t - kr)$ in a propagation factor, so long as the relative sign between the two terms is negative to ensure propagation towards larger r; c.f. discussion problem]

To get the electric fields corresponding to this potential (the outward going kidney shaped loops of lines of force on the previous page) we use (1.12) $\mathbf{E} = -\nabla\phi$, as we did in section 1.8 for the field around a static dipole. The result must fall off with distance at least as fast as the $1/r^2$ factor in (14.7) [in the static case it falls off like $1/r^3$, c.f.. (1.42), (1.43)].

But the fields due to the current $I = I_0 \sin \omega t$ fall off more slowly than $1/r^2$.

Amperes's law requires that the magnetic field due to an infinitely long straight wire carrying steady current I is $B = \dfrac{\mu_0 I}{2\pi r}$.



14.3 Motivation for the Magnetic Vector Potential *A*.

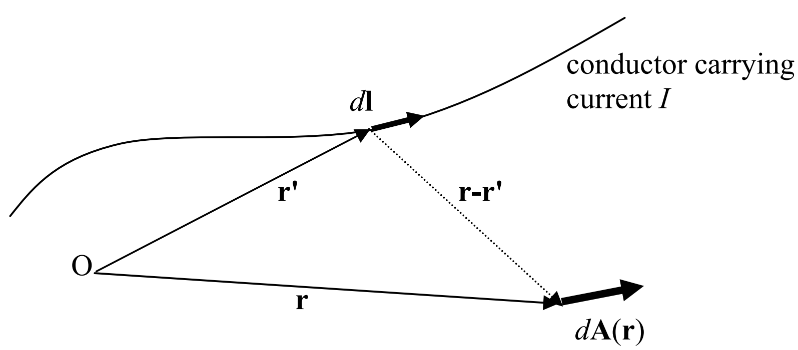
For the short Hertzian dipole we have to integrate the effect over a finite length. That is very clumsy with the Biot Savart law but is quite straightforward using the magnetic vector potential $\mathbf{A}(\mathbf{r},t)$, which is defined by close analogy with the scalar electrical potential $\phi(\mathbf{r},t)$.

We use (1.12) $\mathbf{E} = -\nabla\phi$ to define the potential ϕ in terms of the measurable field **E**, and because this is a differential expression we are left with an ambiguity in the value of ϕ which is dealt with by making an arbitrary choice of the zero reference point. (All voltage measurements are relative – i.e. difference of potential between two points). Similarly, we use the differential expression

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{14.8}$$

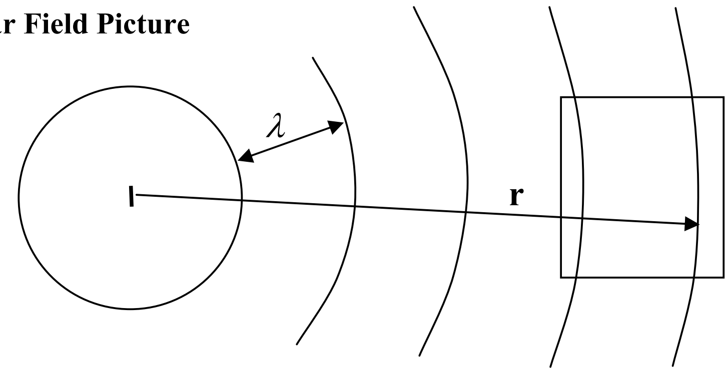
to define **A** in terms of the measurable field **B.** Because it is a vector it has a more complicated set of ambiguities which have to be dealt with by a process called "choosing the gauge" (see Grant and Phillips for a flavour; see Jackson's "Classical Electrodynamics" (Wiley) for the full beauty of it).

The reason **A** is so much easier to use than the Biot Savart law is that $\mathbf{A}(\mathbf{r})$ is just proportional to a sum of all the current densities $d\mathbf{I}(\mathbf{r}')$ which are present, without any cross products in the integral. (The diagram below is taken from section 2 above, with one modification. Note that the contribution $d\mathbf{A}(\mathbf{r})$ is parallel to current element $d\mathbf{I}$ which causes it, while the contribution $d\mathbf{B}(\mathbf{r})$ was at an angle given by a cross-product.)



So in the hertzian dipole $\mathbf{A}(\mathbf{r},t)$ will be proportional to the only current present, $\mathbf{I}(t') = I_0 \hat{\mathbf{l}} \sin \omega t'$, summed over the length of the dipole. Because have used the retarded time the expressions for the components of the vector potential take a very similar shape to equation (14.7) for the scalar potential.

14.4 Far Field Picture



For example (c.f. G and P), in the far field with $r >> \lambda >> l$

$$A_z(\mathbf{r},t) = \frac{I_0 \mu_0 l}{4\pi r} \sin\left(\omega t - \frac{\omega r}{c}\right), \qquad (14.9)$$

where the sinusoidal term is again recognisable as a wave propagation factor. If we assume the dipole is aligned in the *z* direction there are no components of **I** in the *x* or *y* directions, so **A** only has this *z* component.

Note that in (14.9) **A** falls off only like $1/r$. To get the corresponding **B** fields at large **r** we can use (14.8) $\mathbf{B} = \nabla \times \mathbf{A}$. From our Tools handout, in cylindrical polars (not for the calculation, just to illustrate the point)

$$\nabla \times \mathbf{A} = \frac{1}{r} \begin{vmatrix} \hat{\mathbf{r}} & r\underline{\hat{\varphi}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ A_r & rA_\varphi & A_z \end{vmatrix} .$$

We can see that there will only be one nonzero term in this curl, representing the φ component of **B,** since**:**

> on the bottom line only A_z is nonzero, and two of the derivative operators on the middle line can act upon A_z, but it has no φ dependence so only $\partial/\partial r$ contributes and gives two terms; the first from differentiating the amplitude, proportional to $1/r^2$, which falls away faster than the second, proportional to ω/r, which comes from differentiating the propagation factor.

The final expression is (doing the curl in spherical polars which gives a $\sin\theta$ factor, and going back to our exponential notation for the propagation factor)

$$B_\varphi = \frac{i\omega I_0 l \sin\theta}{4\pi\varepsilon_0 c^2} . \frac{\exp i\left(\omega t - \mathbf{k.r}\right)}{cr} \qquad (14.10)$$

Note that because of the $\sin\theta$ term there will be zero amplitude for transmission along the polar direction $\pm z$ but maximal amplitude, falling off only as $1/r$, for transmission in the equatorial plane of the dipole.

If we go out a long way in the far field then locally in a box with sidelengths $<<r$, like the one drawn on the right of the diagram above, the wavefronts of **B** can be regarded locally as nearly flat, and since the only component is in the φ direction it is perpendicular to outward propagation direction **r**. Also at this distance the **E** field due to the electric dipole contribution (the moving kidneys) will have fallen off at least as fast as $1/r^2$. But we know from our discussion of plane electromagnetic waves in section 7 that there is a direct link between the size and direction of the **B** and the size and direction of **E** from the Faraday/Maxwell equation $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ which gave us (7.14) $E_0 = cB_0$ for plane waves in vacuo. Using these arguments with (14.10) we get

$$E_\theta = \frac{i\omega I_0 l \sin\theta}{4\pi\varepsilon_0 c^2} \cdot \frac{\exp i\left(\omega t - \mathbf{k.r}\right)}{r} \tag{14.11}$$

with no other significant components.

There are no r or θ components of **B** because I flows only in the axial direction, up and down the dipole. There is no r component of **E** because in the far field the electrostatic effect has fallen to negligible size. There is no ϕ component of **E** because **E** must be perpendicular to **B**, which only has a ϕ component.

14.5 Energy Flow in the Far Field

The Poynting vector averaged over a whole number of cycles, at radius r from the dipole and at angle θ to its axis is given by

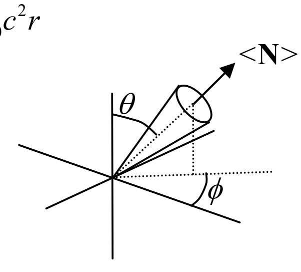
(10.10) $$< N > = \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}} E_0^2 \text{ watts/m}^2$$

where, from (14.11)

$$E_0 = \frac{\omega l I_0 \sin\theta}{4\pi\varepsilon_0 c^2 r} \tag{14.12}$$

Remember that the solid angle subtended by area ds at distance r is $d\Omega = \frac{ds}{r^2}$,

where $\int_{\substack{\text{whole}\\\text{sphere}}} d\Omega = 4\pi$ steradians.



If we choose the element ds in spherical polars then $ds = r^2 d\Omega = r^2 \sin\theta d\theta d\phi$. From (10.10) and (14.12) we can get the total outward flux of energy in all directions, i.e. the power radiated by the dipole

$$P = \int_{\substack{\text{whole}\\\text{sphere}}} < N > ds = \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}} \frac{\omega^2 l^2 I_0^2}{16\pi^2 \varepsilon_0^2 c^4 r^2} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \sin^3\theta d\theta d\phi \text{ watts.} \tag{14.13}$$

The ϕ integration gives a simple factor of 2π.

We need a substitution for the θ integration. Use $\int_{\theta=0}^{\pi} \sin^3 \theta d\theta = -\int_{\theta=0}^{\pi} (1-\cos^2\theta) d\cos\theta$

and $x = \cos\theta$.

Then $-\int_{1}^{-1}\left(1-x^2\right) dx = -\left[x - \frac{x^3}{3}\right]_1^{-1} = \left(1+1-\frac{1}{3}-\frac{1}{3}\right) = \frac{4}{3}$.

Also $\sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{1}{\varepsilon_0^2} = \sqrt{\frac{1}{\varepsilon_0\mu_0}}\frac{1}{\varepsilon_0} = \frac{c}{\varepsilon_0}$.

So substituting back into (14.13), the total power from the dipole is

$$P = \frac{2\pi}{2}\frac{c}{\varepsilon_0}\frac{4}{3}\frac{\omega^2 l^2 I_0^2}{16\pi^2 c^4}.$$

Therefore

$$P = \frac{\omega^2 l^2 I_0^2}{12\pi\varepsilon_0 c^3} \qquad (14.14)$$

Note 1. The power is proportional to ω^2. It is far easier to radiate at high frequencies than at low; in fact it is hard to work with high frequencies (>MHz) without severe crosstalk effects due to radiation of energy from one circuit to another, unless screening and grounding have been done very carefully. Part of the technique is to use transmission lines or waveguides to carry R.F. power because the fields remain localized in the region of the conductor(s) rather than spreading out into space.

Note 2. The factor of $1/r^2$ from E_0^2 in (14.13) exactly cancels the factor of r^2 in the element of area ds. If E_0 fell off any slower than 1/r then the power would grow with radius and energy would not be conserved.