2B29 Electromagnetic Theory

13. Waveguides (and a little about Transmission Lines)

13.1 The Energy is in the Space

We saw in section 11 above that when an electromagnetic wave falls on a medium with high conductivity $\sigma >> \varepsilon_0\omega$ the size of the reflection coefficient at normal incidence is given by equation

(11.11) $$r_n \simeq \frac{-n'}{+n'} = -1,$$

that is, all of the energy is reflected. This can be shown to be true at all angles of incidence. We also showed that the disturbance penetrates only a few skin depths $\delta = \frac{1}{\operatorname{Im} k} = \sqrt{\frac{2}{\mu\omega\sigma}}$ into the material; equation (11.7). For copper at 100 MHz, for instance, we saw that $\delta \simeq 7.1\mu m$.

So when we come to transmit electromagnetic waves at high frequencies in "captive" form through either transmission lines or waveguides, the important energy flow occurs in the spaces between the conductors, not in the thin layers of conduction inside the conductors themselves.

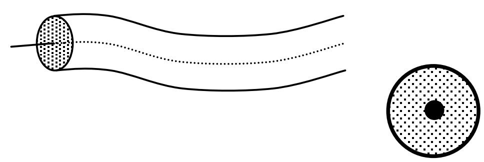
13.2 Transmission Lines

(Not on the syllabus, but so important they need a mention. Ask for a handout if you want to know more)

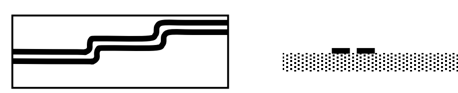
Over a very wide range of frequencies, from audio (10s to thousands of Hz) to UHF TV (~30 MHz) and beyond, we often use transmission lines to transport signals or pulses. They normally have two separate conductors.

Examples:

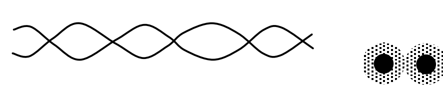
i) Co-axial cable. This has an outer conducting sheath and a conducting wire core, separated by a cylindrical layer of dielectric material.



ii) Stripline. Parallel conductors on a dielectric substrate; on a printed circuit board, for instance or on a flexible kaptan ribbon.



iii) Twisted pair. Two insulated wires twisted around one another in a regular helix.

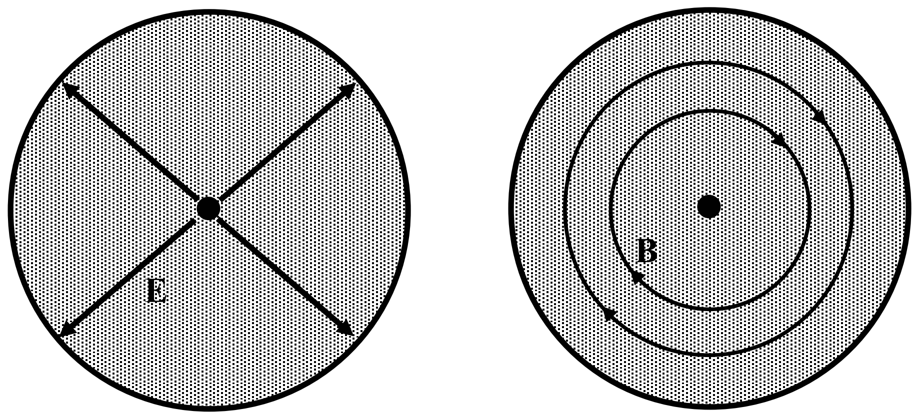


Points to note

a) Any transmission line can be shown to have a characteristic impedance (coax is often 50Ω or 75Ω, twisted pair ~110Ω -depending on the geometry of the conductors and the relative permittivity of the dielectric) If they are not *terminated* with a receiver that looks like a resistor equal to the characteristic impedance there will be severe reflections of pulses and distortions of signals.

b) The waves in a transmission line have both their **E** and **B** vectors perpendicular to the direction of transmission – in a similar way to plane waves in free space. Both are called TEM waves; for Transverse Electrical and Magnetic.

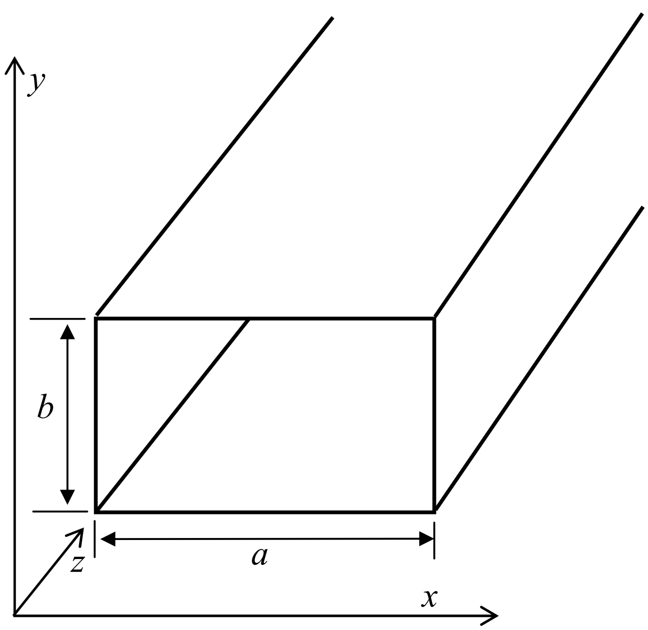
For example in a coaxial cable at some time t there can be a potential difference $V(z,t)$ between the inner and outer conductors at a point a distance z from the end of the cable.



This will give rise to a radial **E** field. At the same time currents *I(z,t)* flowing in opposite directions in the outer sheath and the inner core give rise to concentric lines of **B**, everywhere perpendicular to **E** and to the z direction in which the signal is transmitted.

c) Far from the ends of a transmission line the two conductors are equally important to the high frequency signal. One of them – often the sheath of a coax. - may be the DC ground connection, but this is not relevant to the high frequency signal.

13.3 Simple rectangular Waveguides
Waveguides and RF cavities have only one conductor, for instance a rectangular hollow pipe.



Again, we transmit waves in the z direction.

The fields inside the guide have to satisfy an appropriate set of Maxwell equations; assuming they are in air or in vacuum so there will be no charge density ρ or current density **J.**

We go back to section 6 and collect the simplified equations we need.

We assume air or vacuum, so $\mathbf{D} = \varepsilon_0 \mathbf{E}$ and $\mathbf{H} = \dfrac{\mathbf{B}}{\mu_0}$, so we only need **E** and **B**.

Equation (6.1), Gauss' law, becomes

$$\nabla.\mathbf{E} = 0 . \qquad (13.1)$$

Equation (6.2), Faraday's law , remains

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} . \qquad (13.2)$$

Equation (6.3), "no monopoles, remains

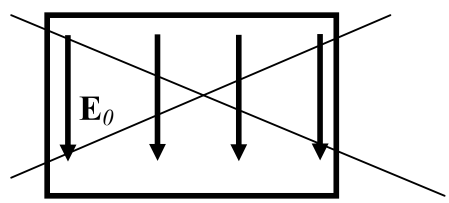
$$\nabla.\mathbf{B} = 0 . \qquad (13.3)$$

Equation (6.4), Ampere's law augmented, becomes

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \qquad (13.4)$$

This is effectively the same simplified set of Maxwell equations we used to derive the nondispersive wave equations (6.19) $\nabla^2 \mathbf{B} = \varepsilon\mu \dfrac{\partial^2 \mathbf{B}}{\partial t^2}$ and (7.3) $\nabla^2 \mathbf{E} = \varepsilon\mu \dfrac{\partial^2 \mathbf{E}}{\partial t^2}$. We have so far dealt mainly with plane wave solutions to these equations, which are required to be TEM by (7.12) $\mathbf{k} \times \mathbf{E}_0 = +\omega \mathbf{B}_0$.

Putting such plane wave TEM solutions into the rectangular waveguide gives immediate difficulties. If $\mathbf{E}_0$ is vertical e.g. in the $-y$ direction, as shown, then if it is finite in the middle of the guide it will also be finite on the surface of the vertical conductors at the edges. But finite E fields parallel to a conducting surface will give significant currents in the metal, which will cause ohmic dissipation of heat and rapidly damp the waves.



Remember, in setting up Poynting's theorem in section 10.1 we noted that the rate of dissipation of energy was proportional to **J.E**. If the current density **J** is along the direction of **E** then work is done by **E** on the moving charges. This imposes a rigorous *boundary condition* for waves to be transmitted without dissipation inside a waveguide. *The component of* **E**(**r**,t) *parallel to any conducting surface must be zero at that surface.* This means that simple TEM waves cannot be transmitted in a waveguide with a single conductor (why the diagram above is crossed out!).

The classes of wave which can be transmitted in a guide are called

TM = Transverse Magnetic

and TE = Transverse Electric

with the implication that for TM the **E** vector may have both transverse and longitudinal components, and for TE the **B** vector may have both transverse and longitudinal components.

13.4 Solving Maxwell's Equations for TM Waves.

TM implies $B_z = 0$, but $E_x(x,y,z,t)$, $E_y(x,y,z,t)$, $E_z(x,y,z,t)$, $B_x(x,y,z,t)$ and $B_y(x,y,z,t)$ may be finite. Start by allowing

$$E_z = E_{z0} \exp i\left(k_g z - \omega t\right) \neq 0 \tag{13.5}$$

where the guide-wavenumber

$$k_g = \frac{2\pi}{\lambda_g}, \tag{13.6}$$

in contrast with the TEM free space wavenumber

$$k_0 = \frac{2\pi}{\lambda_0} = \frac{\omega}{c}. \tag{13.7}$$

If the wave is undamped then E_{z0} stays constant with *z* as the wave propagates **BUT** we can allow it to vary according to the lateral position inside the guide. This enables us to satisfy the boundary condition at the surfaces of the conductor.

$$E_{z0}(x,y) = 0, \text{ for } x = 0 \text{ or } x = a,\ y = 0 \text{ or } y = b, \tag{13.8}$$

The ∇^2 and $\frac{\partial}{\partial t}$ operators in equation (7.3) $\nabla^2 \mathbf{E} = \varepsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2}$ apply independently to the three components of **E**, so we get

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} \tag{13.9}$$

with similar equations for E_x and E_y.

Putting (13.5) into (13.9) we have

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} - k_g^2 E_z = \frac{-\omega^2}{c^2} E_z = -k_0^2 E_z$$

which can be rearranged, with cancellation of the propagation factor $\exp i(k_g z - \omega t)$,

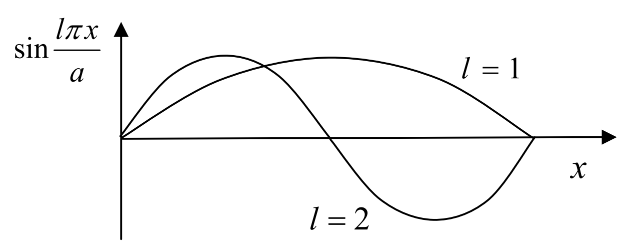
to get

$$\frac{\partial^2 E_{z0}}{\partial x^2} + \frac{\partial^2 E_{z0}}{\partial y^2} = (k_g^2 - k_0^2) E_{z0} \tag{13.10}$$

This equation for $E_{z0}(x,y)$ has to be solved with the boundary conditions of (13.8). It is the equation for the amplitude of standing waves on any 2-dimensional surface, for instance the normal vibration modes of a rectangular drumhead. The solution has the form

$$E_{z0} = E_0 \sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) \tag{13.11}$$

where *l* and *m* are integers and E_0 is the peak value of E_{z0}. The higher the value of *l* or *m*, the more zeros there are as a function of *x* or *y* in the standing wave amplitude. For TM modes *l* or $m = 0$ do not satisfy (13.8) since for finite E_0 they would give finite E_{z0} at some of the surfaces.



Putting (13.11) in (13.10) gives

$$\frac{l^2}{a^2}+\frac{m^2}{b^2}=\frac{k_0^2-k_g^2}{\pi^2}\text{; the } \textit{waveguide equation.} \tag{13.12}$$

It is a sort of dispersion relation since k_g is the wavenumber for propagation along the guide and $k_0=\frac{\omega}{c}$, so (13.12) can be transformed to relate k_g to ω as a dispersion relation should:

$$\omega^2=c^2\left(k_g^2+\pi^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}\right)\right). \tag{13.13}$$

Hence $2\omega\frac{\partial\omega}{\partial k_g}=2c^2k_g$, or $\frac{\omega}{k_g}\frac{\partial\omega}{\partial k_g}=c^2$, so $v_{\text{phase}}v_{\text{group}}=c^2$, as for the plasma in Case 1 of section 12.3 above.

Each TM_{lm} solution represents a different mode of propagation in the waveguide. Starting from (13.5) and (13.11) we can write

$$E_z=E_0\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\exp i\left(k_gz-\omega t\right) \tag{13.14}$$

This can be used with Maxwell's equations (13.1) to (13.4) to get explicit expressions for the other nonzero components $E_x(x,y,z,t)$, $E_y(x,y,z,t)$, $B_x(x,y,z,t)$ and $B_y(x,y,z,t)$. **E** and **B** are perpendicular at all points and times.

13.5 The Cutoff Frequency and Wavelength

The waveguide equation (13.12) can be put in terms of wavelengths since for any wavenumber $k=\frac{2\pi}{\lambda}$, i.e. $\frac{1}{\lambda_g^2}=\frac{1}{\lambda_o^2}-\frac{1}{4}\left(\frac{\ell^2}{a^2}+\frac{m^2}{b^2}\right)$

We can then define a *cutoff wavelength* λ_C given by $\frac{1}{\lambda_C^2}=\frac{1}{4}\left(\frac{\ell^2}{a^2}+\frac{m^2}{b^2}\right)$ so that

$$\frac{1}{\lambda_g^2}=\frac{1}{\lambda_o^2}-\frac{1}{\lambda_C^2} \tag{13.15}$$

i.e. if $\lambda_O=\lambda_C$ then $\lambda_g\to\infty$ and if $\lambda_O>\lambda_C$ then λ_g does not exist and propagation down the wave guide cannot occur. Only modes for which $\lambda_O<\lambda_C$ will be propagated.

There is an equivalent *cutoff frequency*

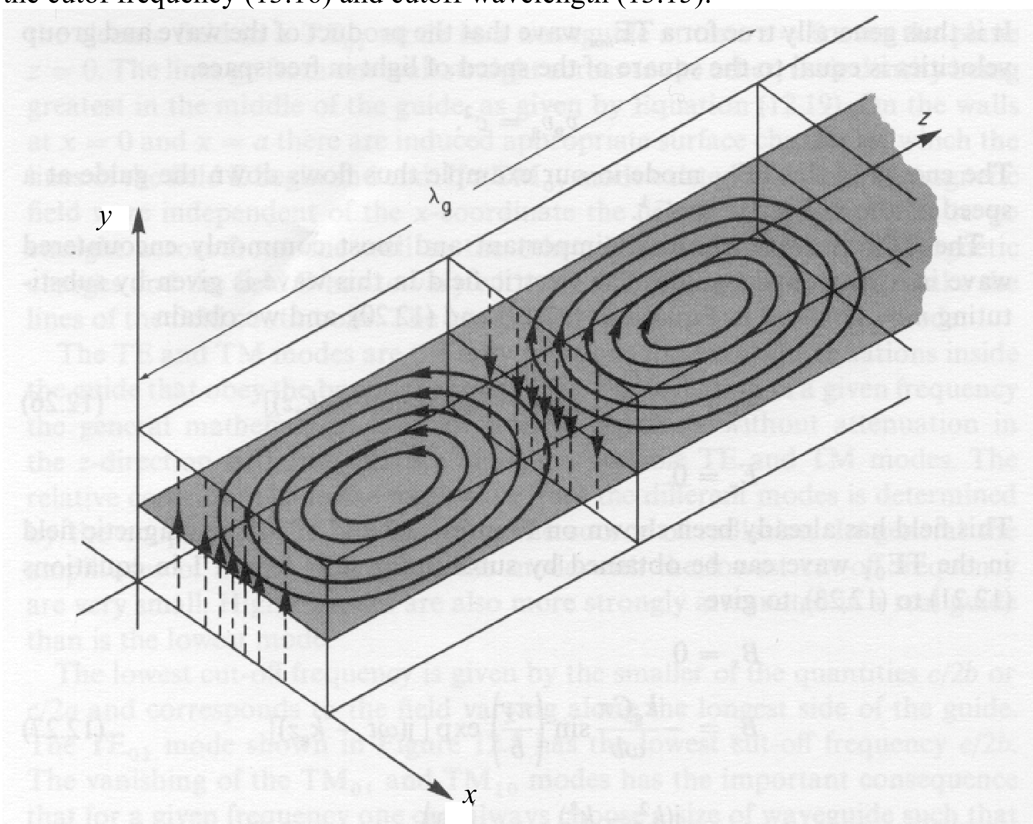
$$\nu_c=\frac{c}{\lambda_c}=c\sqrt{\left(\frac{l}{2a}\right)^2+\left(\frac{m}{2b}\right)^2} \tag{13.16}$$

below which the guide cannot transmit in the TM_{lm} mode.

13.6 TE_{lm} Modes of a Rectangular Waveguide.

By an equivalent strategy to section 13.4 above we can also derive the nonzero components $E_x(x,y,z,t)$, $E_y(x,y,z,t)$, $B_z(x,y,z,t)$, $B_x(x,y,z,t)$ and $B_y(x,y,z,t)$ of the TE waves. See Grant and Phillips for explicit expressions (with different definitions of axes!). The standing wave patterns in *x* and *y* are similar to (13.14), so they are also labeled by *l* and *m* mode numbers which can be shown to satisfy the same waveguide equation (13.12). There is one important difference. Either *l* or *m* may go to zero, but not both.

Since the TE_{lm} modes obey the waveguide equation they have the same equations for the cutof frequency (13.16) and cutoff wavelength (13.15).



This picture from Grant and Phillips shows the magnetic field lines (solid) in the TE_{10} mode at an instant of time when the electric field (shown dotted) is a maximum in the +*y* direction over the plane $z = 0$. The maximum reversed electric field is also shown on the plane at $z = \lambda_g/2$. There are finite electric fields at places on the upper and lower surfaces, but **E** is perpendicular to the surface there so the boundary condition is satisfied. The magnetic field lines in this mode have the same shape at all *y*.

The TE_{10} or TE_{01} modes are particularly important because one or the other of them has the lowest cutoff frequency of all the TE_{lm} and TM_{lm} modes (depending on whether *a*>*b* or vice versa). Rectangular waveguides are normally constructed for use with a generator of a particular frequency ν, such that $\nu > \nu_c$ for TE_{10} (or TE_{01}), but $\nu < \nu_c$ for all higher TE and TM modes, including TE_{01} (or TE_{10}).

For example, radar waves with $\nu = 10GHz = 10^{10}\,Hz$ have a free space wavelength of 30mm. We may transport them to the dish-aerial through a waveguide with lateral dimensions $a = 10mm$ and $b = 20mm$. This has cutoff frequencies given by

(13.16) $$\nu_c = c\sqrt{\left(\frac{l}{2a}\right)^2 + \left(\frac{m}{2b}\right)^2}$$

i.e. for TE_{01}, $\nu_c = c\sqrt{\left(\frac{1}{2b}\right)^2} = \frac{c}{2b} = \frac{3\times10^8}{2\times20\times10^{-3}} = 7.5\times10^9\,Hz$

for TE_{10} $\quad \nu_c = \frac{3\times10^8}{2\times10\times10^{-3}} = 1.5\times10^{10}\,Hz$

for TE_{11} and TM_{11} $\quad \nu_c = c\sqrt{\left(\frac{1}{20}\right)^2 + \left(\frac{1}{40}\right)^2} = \frac{c}{40}\sqrt{5} = 0.167\times10^{11} = 1.67\times10^{10}\,Hz$.

Higher modes have higher cutoff frequencies.

So the input signal can only be transmitted in the TE_{01} mode.