2B29 Electromagnetic Theory

12. Waves in Conducting Media, part 2

12.1 Properties of a neutral Plasma

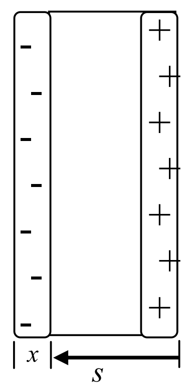
(A very rich and far-ranging subject. We touch on the most basic aspects)

A plasma can be regarded as a matrix of slowly moving massive positive ions surrounded by a free swarm of electrons, N_e per unit volume. There are no localised regions of –ve or +ve charge excess, except where induced for very short times by the local effects of the electromagnetic fields. In the absence of any fields each electron may be regarded as close to a neutral position among the surrounding ions. If it, and its local neighbours are all displaced by a distance **x** then there will be a local polarisation density

$$\mathbf{P} = -N_e e\mathbf{x} \tag{12.1}$$

which will generate a local **E** field which gives a restoring force proportional to the displacement, pushing the electrons back towards their neutral position. This naturally gives rise to simple harmonic oscillations of the electron cloud, with a frequency called the *plasma frequency* ω_p.

Derivation of the plasma frequency (rev 1/3/03)



All the electrons in a thin slab of width s are displaced by small distance $x << s$. This creates a negative surface charge on the left hand surface, and leaves a positive charge on the right hand surface. It is like a parallel plate capacitor with charge $Q = \pm N_e exA$ on its plates, each with area A. Assume (for now only) that the plasma has such low density that $\varepsilon_r \simeq 1$.

From 1B26 (or section 1.4 above) the electric displacement field in the gap is $D = \dfrac{Q}{A}$, hence $E = \dfrac{Q}{\varepsilon_0 A} = \dfrac{N_e exA}{\varepsilon_0 A} = \dfrac{N_e ex}{\varepsilon_0}$; so the force on an individual electron, mass m, is $-eE = -\dfrac{N_e e^2 x}{\varepsilon_0}$, giving an equation of motion $\dfrac{d^2x}{dt^2} = -\dfrac{N_e e^2 x}{\varepsilon_0 m}$; the simple harmonic motion equation with solution

$x(t) = X \exp i\omega_p t$, and the plasma frequency is $\omega_p \equiv \sqrt{\dfrac{N_e e^2}{\varepsilon_0 m}}$ (12.2)]

12.2 Electromagnetic Waves in a Plasma

Collisions between electrons and ions are relatively infrequent. If a high frequency wave is passing through the plasma we can regard the electrons as reacting freely to the instantaneous electric and magnetic effects of the wave, at least for the duration of a few cycles of oscillation. In contrast, in a metal any energy given to an electron is immediately dissipated by ohmic collisions so we write $\mathbf{J} = \sigma\mathbf{E}$ as a good approximation.

Let us assume a plane electromagnetic wave of the usual form $\mathbf{E} = \mathbf{E}_0 \exp i\left(\mathbf{k.r} - \omega t\right)$. We use the Lorentz force (1.1) to get the force on an electron

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = -e\left(\mathbf{E}(\mathbf{r},t) + \mathbf{v}\times\mathbf{B}(\mathbf{r},t)\right) \qquad (12.3)$$

In fact, the magnetic contribution is small in a nonrelativistic plasma. Remember from (7.14) (or from the Faraday Maxwell equation) $B_0 = \frac{E_0}{\mathrm{v_p}}$, so the ratio of the two parts of the right hand side of (12.3) is $\frac{|\mathbf{v}\times\mathbf{B}|}{|\mathbf{E}|} \approx \frac{|\mathrm{v}|}{\mathrm{v}_p}$. But $\mathrm{v}_p \approx c >> \mathrm{v}$ unless something has accelerated the electrons to relativistic speeds, so this magnetic term from the Lorentz force can be neglected.

Equation (12.3) then simplifies to

$$m\frac{d\mathbf{v}}{dt} = -e\mathbf{E} = -e\mathbf{E}_0 \exp i(\mathbf{k.r} - \omega t),$$

which can be integrated over time

$$\int \frac{d\mathbf{v}}{dt}dt = -\frac{e}{m}\mathbf{E}_0 \exp(i\mathbf{k.r})\int \exp\left(-i\omega t\right)dt$$

$$\mathbf{v} = -i\frac{e\mathbf{E}}{m\omega} \qquad (12.4)$$

The collective velocity of all the electrons constitutes a current

$$\mathbf{J} = -N_e e\mathbf{v} = i\left(\frac{N_e e^2}{m\omega}\right)\mathbf{E} \qquad (12.5)$$

The factor i means that the $\mathbf{J}(\mathbf{r},t)$ is $\pi/2$ out of phase with the driving $\mathbf{E}$ field – quite different from a metal where $\mathbf{J} = \sigma\mathbf{E}$ and a real conductivity σ means that $\mathbf{J}$ is in phase with $\mathbf{E}$.

To avoid developing lots of formal plasma theory, we again go back to the picture of the individual electrons in some small region displaced by $\mathbf{x}$ from their natural neutral position; closely analogous to the picture (sections 1.3 and 1.4 above) of electrons in a dielectric which we used to derive the basic definitions of the electric displacement.

(1.26) $$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon_r\varepsilon_0\mathbf{E}.$$

If we take $\mathbf{P} = -N_e e\mathbf{x}$, comparing with (12.5) we have

$$\mathbf{J} = -N_e e\mathbf{v} = \frac{\partial\mathbf{P}}{\partial t} \qquad (12.6)$$

This is the contribution to the displacement current in the plasma from the movement of electrons giving a change of polarisation with time. There will also be a contribution which would be present even in vacuum, from the change of **E** with time, so we can write the whole of the displacement current as

$$\frac{\partial \mathbf{D}}{\partial t} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} = \varepsilon_r \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Putting in the plane wave expression for $\mathbf{E}(\mathbf{r},t)$ and (12.5) we get

$$-\varepsilon_0 i\omega \mathbf{E} + i\left(\frac{N_e e^2}{m\omega}\right)\mathbf{E} = -\varepsilon_r \varepsilon_0 i\omega \mathbf{E}$$

Or

$$\varepsilon_r = 1 - \frac{1}{\omega^2}\frac{N_e e^2}{\varepsilon_0 m} = 1 - \frac{\omega_p^2}{\omega^2} \tag{12.7}$$

Note that, from $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_r \varepsilon_0 \mathbf{E}$ (1.26), if a wave has finite displacement amplitude D_0, the amplitude of the response E_0 of the plasma will be proportional to D_0 / ε_r. But if $\omega \to \omega_p$, (12.7) requires that $\varepsilon_r \to 0$, so $E_0 \to \infty$ to maintain finite D_0, and the amplitude of the oscillations in polarization $P_0 \to \infty$ also. This means the plasma electrons are behaving like an undamped harmonic oscillator close to its resonant frequency ω_p. As always in such situations, the amplitude of oscillations grows until the system changes in some way. A mechanical system may just break. Most systems become non linear and begin to dissipate energy in ways which are not described by the simple wave equation that works for small amplitude. In a plasma the electrons may get enough speed that they can further ionise or excite the atoms, dissipating the incoming energy. The result is that the wave is strongly absorbed and not transmitted through any region where $\omega \to \omega_p$. In fact, all of the approximations on which the derivation of the plasma frequency and equation (12.7) depend will break down when $\omega \to \omega_p$.

The phase velocity is $\mathrm{v_p} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} \simeq \frac{1}{\sqrt{\mu_0 \varepsilon_0 \varepsilon_r}}$ so, if $\mu \simeq \mu_0$ in plasma, we get $k^2 = \omega^2 \varepsilon_0 \varepsilon_r \mu_0 = \frac{\omega^2 \varepsilon_r}{c^2}$. Substituting from (12.7) we get

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right) \tag{12.8}$$

This is a dispersion relation for waves in the plasma.

12.3 Applications of the Plasma Dispersion relation

<u>Case 1</u>, $\omega > \omega_p$. From (12.8) $k^2 > 0$ so k is real and the waves pass through the plasma unattenuated. The phase and group velocities obey an interesting relation. Remember $\mathrm{v_p} = \frac{\omega}{k}$, $\mathrm{v_g} = \frac{\partial \omega}{\partial k}$. Transforming (12.8) we get $\omega^2 = \omega_p^2 + k^2 c^2$, so differentiating gives $2\omega \frac{\partial \omega}{\partial k} = 2kc^2$ which can be rewritten $\mathrm{v}_g = \frac{k}{\omega} c^2 = \frac{c^2}{\mathrm{v}_p}$, or as

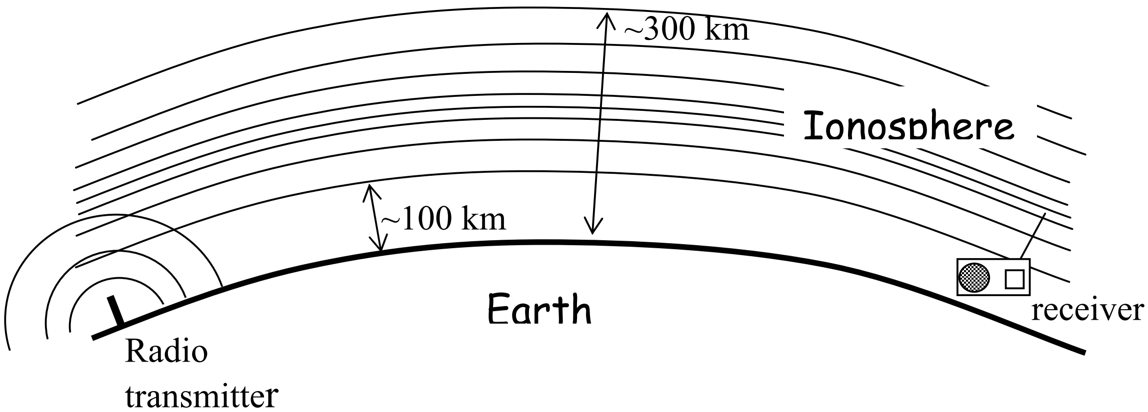
$$\mathrm{v_p v_g} = c^2 . \tag{12.9}$$

So one of the two velocities must be $>c$ if either of them is $\neq c$. To find which velocity is superluminal we remember that $n = \dfrac{c}{\mathrm{v}_p}$, so $n^2 = \dfrac{c^2k^2}{\omega^2} = \left(1 - \dfrac{\omega_p^2}{\omega^2}\right)$ and n is therefore real and <1. Hence $\mathrm{v}_p > c$. (12.10)

This does not violate the predictions of special relativity theory because, when $\mathrm{v}_p > c$, equation (12.9) assures us that $\mathrm{v}_g < c$, and both information and energy travel with the group velocity which is no faster than the velocity of light.

<u>Case 2</u> $\omega < \omega_p$. Equation (12.8) requires k^2 negative, so $\mathbf{k} = \dfrac{i\hat{\mathbf{k}}}{L}$ where L is an attenuation length. As usual imaginary **k** in the propagation factor $\exp i(\mathbf{k.r} - \omega t)$ of a wave gives rise to absorption of the energy – damping of the oscillation – proportional to $\exp\left\{-\dfrac{\hat{\mathbf{k}}.\mathbf{r}}{L}\right\} \times \exp i\omega t$. This predicts no transmission of frequencies below ω_p.

12.4 Radio Waves in the Ionosphere (an introduction)



Energy sources such as ultraviolet radiation from the sun ionise the upper atmosphere. Between 100 and 300 km height the plasma has up to ~10^{11} electrons/m^3, varying with height, time of day or night, time of year, terrestrial weather and solar weather (sunspots, solar wind, magnetic storms etc.). The typical frequency in Hz corresponding to the plasma frequency at the densest part of the ionosphere is given by (12.2)

$$\nu = \frac{\omega_{p\max}}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{N_{e\max}e^2}{\varepsilon_0 m}}$$

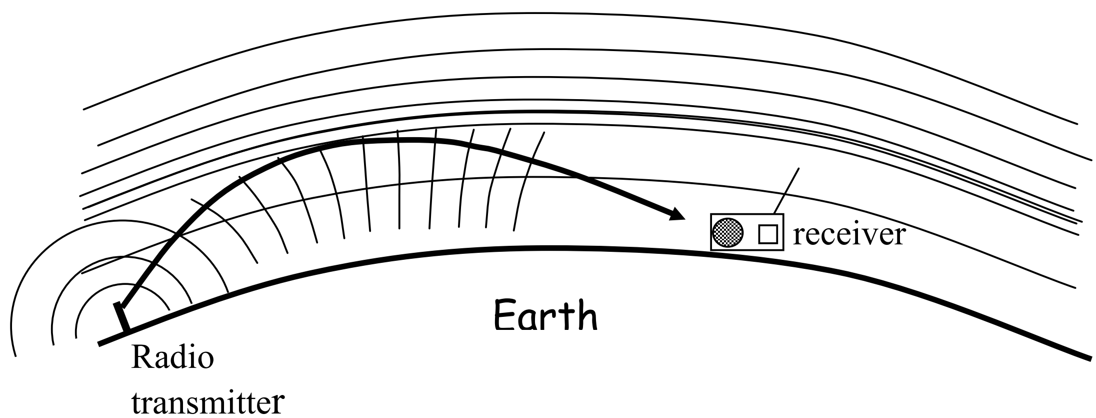
$$= \frac{1.6\times10^{-19}}{2\pi}\sqrt{\frac{10^{11}}{8.85\times10^{-12}\ 9.11\times10^{-31}}} = \frac{0.8\times10^{-19+27}}{\pi\sqrt{8.85\times9.11}} \simeq 2.8\times10^6$$

i.e. about 3 MHz.

A. Radio waves with $\omega > \omega_{p\max}$ there is no absorption. The waves behave as in Case 1 above and satisfy equation (12.9). Transmissions go through the ionosphere and out into space.

GOOD for communication with satellites (EHF, SHF, UHF)
BAD for long range terrestrial broadcasting (VHF>30 MHz) because the waves do not reach receivers beyond the horizon of th transmitter.

B. Radio waves with $\omega \approx \omega_{p\max}$ (HF = "short wave", 3 to 30 MHz, λ = 10 to 100m). If *directly upward* they will escape if N_e is low at the time, or they will be absorbed when they reach the height where $\omega = \omega_p < \omega_{p\max}$, since $\omega_p \propto \sqrt{N_e}$ rises as they enter the ionosphere from below.
But if they are at an *oblique angle* (see figure showing portions of wavefronts at heights below the maximum of N_e) they start off in the normal atmosphere



with $\mathrm{v_p} = c$. The upper parts of the wavefronts then penetrate the ionosphere and equation (12.10) applies. The diagram shows how the sections of the wavefronts in the denser part of the ionosphere move faster than those below, so surfaces of equal phase are refracted back down towards the earth's surface, over the horizon. This effect is very variable in time and place, which explains why short wave communications keep cutting out or even give double signals. (You cannot rely on getting the BBC World Service in e.g. Geneva at the same frequency every day!)

C. Radio waves with $\omega < \omega_{p\max}$ are always trapped below the ionosphere. Again, components transmitted steeply upward are strongly absorbed. Components transmitted close to the horizontal direction are reflected by the underside of the ionosphere (as in B above) and scattered from features on the surface of the earth. MW may be detected over ~100 kilometres. LW (e.g. λ=1500m, ν=198 kHz, Radio 4) can be detected at up to ~1000 km or more (-sometimes OK in Geneva, but just behind the Jura mountains which cast a shadow).