2B29 Electromagnetic Theory

10. Energy Flow and the Poynting Vector

10.1 Energy Flow and Dissipation in Waves

We saw in section 4 that the energy per unit volume stored in an electrostatic field is given by the expressions (renumbered here for ease of reference)

(4.3) $$U_e = \frac{1}{2}\mathbf{E.D} \text{ Joules/m}^3 \qquad (10.1)$$

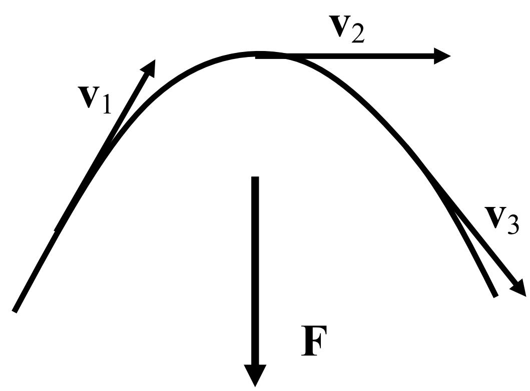
and the energy density in a magnetic field by

(4.9) $$U_m = \frac{1}{2}\mathbf{H.B} \text{ Joules/m}^3 \qquad (10.2)$$

Now we want to think about the way energy moves in an electromagnetic wave; not just the plane waves we have been using for the discussion of optics, but any form of general electromagnetic wave. To understand the problem we take a situation in which some electromagnetic energy is being taken away from the fields and dissipated by a current density $\mathbf{J}(\mathbf{r},t)$ flowing in a resistive medium. The power being taken from the field and transferred to the current per unit volume will be $\mathbf{J.E}$.

Note that if $\mathbf{J} \perp \mathbf{E}$ no energy is being transferred since the current will be moving along an electric equipotential.

Compare with. a projectile moving along a parabolic trajectory under a gravitational force $\mathbf{F} = mg\hat{\mathbf{f}}$. The rate of transfer of energy from the potential energy in the field to the kinetic energy of the body is $\mathbf{v.F}$. On the diagram the gravitational potential is increasing where $\mathbf{v}_1.\mathbf{F} < 0$ and is decreasing where $\mathbf{v}_3.\mathbf{F} > 0$. When the projectile is moving horizontally with velocity $\mathbf{v}_2$ no energy is being transferred because $\mathbf{F} \perp \mathbf{v}_2$; the projectile is moving along a gravitational equipotential.]



Rate of transfer of energy from field to the current in volume τ is, in general,

$$P_\tau = \int_\tau \mathbf{J.E}\, d\tau \qquad (10.3)$$

In the special case where the current $\mathbf{J}$ is caused only by the $\mathbf{E}$ field acting upon the medium with conductivity σ then

$$P_\tau = \int_\tau \sigma E^2 d\tau \qquad (10.4)$$

We now use Maxwell's equations (6.1) to (6.4)to transform (10.3). From the Ampere law (6.4) $$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

so $$\mathbf{J} = \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t}$$

then (10.3) becomes $$\int_\tau \mathbf{J}.\mathbf{E}d\tau = \int_\tau \left\{ \mathbf{E}.(\nabla \times \mathbf{H}) - \mathbf{E}.\frac{\partial \mathbf{D}}{\partial t} \right\} d\tau .$$

Our mathematical tools tell us that
$$\nabla.(\mathbf{E} \times \mathbf{H}) = \mathbf{H}.(\nabla \times \mathbf{E}) - \mathbf{E}.(\nabla \times \mathbf{H})$$

so $$\int_\tau \mathbf{J}.\mathbf{E}d\tau = \int_\tau \left\{ \mathbf{H}.(\nabla \times \mathbf{E}) - \nabla.(\mathbf{E} \times \mathbf{H}) - \mathbf{E}.\frac{\partial \mathbf{D}}{\partial t} \right\} d\tau .$$

Substituting from the Faraday law (6.2) $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ we get

$$\int_\tau \mathbf{J}.\mathbf{E}d\tau = -\int_\tau \left\{ \nabla.(\mathbf{E} \times \mathbf{H}) + \mathbf{H}.\frac{\partial \mathbf{B}}{\partial t} + \mathbf{E}.\frac{\partial \mathbf{D}}{\partial t} \right\} d\tau . \tag{10.5}$$

In a linear medium $\mathbf{B} = \mu \mathbf{H}$ and $\mathbf{D} = \varepsilon \mathbf{E}$. We can use that (without a fully general proof) to motivate, from (10.1) and (10.2),

$$\mathbf{H}.\frac{\partial \mathbf{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\mathbf{H}.\mathbf{B} = \frac{\partial}{\partial t}U_m$$

and $$\mathbf{E}.\frac{\partial \mathbf{D}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\mathbf{E}.\mathbf{D} = \frac{\partial}{\partial t}U_e .$$

Putting these results into (10.5), and rearranging, we get

$$-\int_\tau \nabla.(\mathbf{E} \times \mathbf{H}) d\tau = \frac{\partial}{\partial t}\int_\tau \frac{1}{2}\{\mathbf{H}.\mathbf{B} + \mathbf{E}.\mathbf{D}\} d\tau + \int_\tau \mathbf{J}.\mathbf{E}d\tau \tag{10.6}$$

The two terms on the right hand side represent the rate of increase of the stored energy $\frac{1}{2}\{\mathbf{H}.\mathbf{B} + \mathbf{E}.\mathbf{D}\}$ in volume τ added to the rate of dissipation of energy due to the work done by the **E** field on currents **J**(**r**,t) in τ. The left hand side of (10.6) is the volume integral of a divergence.

10.2 The Poynting Vector and Poynting's Theorem

We define the *Poynting vector* $$\mathbf{N} \equiv \mathbf{E} \times \mathbf{H} . \tag{10.7}$$

Then Gauss' theorem requires that $\int_\tau \nabla.\mathbf{N}d\tau = \oint_S \mathbf{N}.d\mathbf{S}$, where $\mathbf{N}.d\mathbf{S}$ is the outward flux of **N** through an element d**S** of the surface S surrounding volume τ.

Poynting's theorem asserts that $\oint_S \mathbf{N}.d\mathbf{S}$ over any closed surface S is the rate of flow of energy through that surface in the form of electromagnetic waves. Rewriting

(10.6) we get $$-\oint_S \mathbf{N}.d\mathbf{S} = \frac{\partial}{\partial t}\int_\tau \frac{1}{2}\{\mathbf{H}.\mathbf{B} + \mathbf{E}.\mathbf{D}\} d\tau + \int_\tau \mathbf{J}.\mathbf{E}d\tau , \tag{10.8}$$

which says that the inward flux of electromagnetic energy into volume τ is equal to the sum of the rate of increase of stored energy in the fields plus the rate of ohmic dissipation due to currents driven by the fields. It is very hard to prove Poynting's theorem in total generality, but the above argument based on (10.8) provides a firm basis for accepting it, and using the Poynting vector $\mathbf{N}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t) \times \mathbf{H}(\mathbf{r},t)$ to represent the energy flow (watts/m^2) at spacetime point $(\mathbf{r},t)$ in an electromagnetic wave.

For an unbounded plane wave of the usual form $\mathbf{E} = \mathbf{E}_0 \exp i(\mathbf{k.r} - \omega t)$ we know from (7.12), or (8.8) and (8.9), $\mathbf{H} = \dfrac{\mathbf{k} \times \mathbf{E}}{\mu\omega}$ and $\dfrac{k}{\omega} = \dfrac{1}{\mathrm{v_p}} = \sqrt{\varepsilon\mu}$. So the *instantaneous* energy flow in the wave is

$$\mathbf{N}(\mathbf{r},t) = \mathrm{Re}\left(\mathbf{E}(\mathbf{r},t)\right) \times \mathrm{Re}\left(\mathbf{H}(\mathbf{r},t)\right) = \sqrt{\frac{\varepsilon}{\mu}}\mathbf{E}_0 \times \left(\hat{\mathbf{k}} \times \mathbf{E}_0\right)\cos^2(\mathbf{k.r} - \omega t).$$

But $\mathbf{E}_0 \times \left(\hat{\mathbf{k}} \times \mathbf{E}_0\right) = E_0^2 \hat{\mathbf{k}}$ (convince yourself!) so

$$\mathbf{N}(\mathbf{r},t) = \sqrt{\frac{\varepsilon}{\mu}} E_0^2 \, \hat{\mathbf{k}} \cos^2(\mathbf{k.r} - \omega t). \tag{10.9}$$

This is a vector in the direction $\hat{\mathbf{k}}$ of motion of the wavefronts, but it varies with time. Often it is more useful to use the time-averaged energy flow

$$< \mathbf{N} >= \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}} E_0^2 \, \hat{\mathbf{k}} \tag{10.10}$$

[Remember; $< \cos^2\theta > = \dfrac{1}{2}$ if the average is taken over a whole number of cycles.]

Equation (10.10) can be rewritten for complex $\mathbf{E}(\mathbf{r},t), \mathbf{H}(\mathbf{r},t)$ as

$$< \mathbf{N} >= \frac{1}{2}\mathrm{Re}\left(\mathbf{E} \times \mathbf{H}^*\right) \ \text{watts}/m^2 \tag{10.11}$$

[You should check for yourself that they are equivalent]

10.3 Pressure due to Electromagnetic Waves

The pressure from a wave can be calculated classically, but we are going to cheat. Since Einstein we know about photons, so it is much easier to assume that all electromagnetic waves are carried by them – even though in a radio wave (say) the individual photons are so feeble and their phases are so coherent that they can be described with great precision as if they were blended together into a single entity represented by the classical $\mathbf{E}(\mathbf{r},t), \mathbf{H}(\mathbf{r},t)$ fields, etc.

We know that photons have invariant mass $m_0 = 0$ and energy $E = \hbar\omega = h\nu$. In special relativity $E^2 = p^2c^2 + m_0^2c^4$, so the momentum of one photon is

$$p = \frac{E}{c} = \frac{h\nu}{c}.$$

So the momentum carried per unit area per second in a plane electromagnetic wave in free space is

$$p_{wave} = \sum_i p_i = \frac{1}{c}\sum_i h\nu_i = \frac{<N>}{c} \qquad (10.12)$$

where i runs over all photons in the 1 m^2 of the wave over a period of 1 second.

If, for example, this wave is totally reflected back at a metallic boundary (next section) then the change of momentum per m^2 in one second is, from (10.10),

$$\Delta p = 2\frac{<N>}{c} = \varepsilon_0 E_0^2 . \qquad (10.13)$$

This is equal to the *pressure on the surface* [you should check dimensions!]

If the wave is totally absorbed without reflection then the momentum transfer is halved and

$$\text{Pressure} = \frac{<N>}{c} = \frac{\varepsilon_0 E_0^2}{2} \text{ Newtons}/m^2 \qquad (10.14)$$