2B29 Electromagnetic Theory

9. Optics; applying the Fresnel Relations

9.1 Fresnel Relations and Energy

Gathering the Fresnel relations together:

(8.15) $$r_{\|} \equiv \frac{E_{0\|}''}{E_{0\|}} = \frac{\frac{n'}{n}\cos\alpha - \cos\alpha'}{\frac{n'}{n}\cos\alpha + \cos\alpha'}$$

(8.17) $$t_{\|} \equiv \frac{E_{0\|}'}{E_{0\|}} = \frac{2\cos\alpha}{\left(\frac{n'}{n}\right)\cos\alpha + \cos\alpha'}$$

(8.19) $$r_{\perp} \equiv \frac{E_{0\perp}''}{E_{0\perp}} = \frac{\cos\alpha - \frac{n'}{n}\cos\alpha'}{\cos\alpha + \frac{n'}{n}\cos\alpha'}$$

(8.21) $$t_{\perp} \equiv \frac{E_{0\perp}'}{E_{0\perp}} = \frac{2\cos\alpha}{\cos\alpha + \frac{n'}{n}\cos\alpha'}.$$

These are all ratios of amplitudes. We will show later that the power carried by a wave – the energy flow per second – is proportional to $\mathbf{E}\times\mathbf{H}$. But, from (2.16) or (8.9), $\mathbf{H}(\mathbf{r},t) = \frac{\mathbf{B}(\mathbf{r},t)}{\mu} = \sqrt{\frac{\varepsilon}{\mu}}\,\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$; i.e. $H = \sqrt{\frac{\varepsilon}{\mu}}E$. So energy flow is proportional to $\sqrt{\frac{\varepsilon}{\mu}}E^2$. This means that the reflection and transmission coefficients in the Fresnel relations are proportional to the square root of the power reflected or transmitted.

9.2 Brewster Angle

At some incident angle α, for any smooth dielectric surface, the top line of (8.15) will go to zero so there is no reflection of waves with the **E** vector in the plane of incidence. All the power of the incident $\mathbf{E}_{\|}$ wave at this angle goes into the refracted wave. But if the incident radiation also has an $\mathbf{E}_{\perp}$ polarisation component (true for any kind of incident waves which are not purely polarised in the plane) then (8.19) shows there will still be some reflection of the $\mathbf{E}_{\perp}$ part. The reflected radiation at this special angle – the Brewster angle - is purely polarised in the $\perp$ direction.

To calculate the Brewster angle α_B we set the top line of (8.15) to zero

$$\frac{n'}{n}\cos\alpha_B - \cos\alpha' = 0, \text{ or}$$

$$n'\cos\alpha_B = n\cos\alpha',$$

With Snell's law (8.6)

$$n\sin\alpha_B = n'\sin\alpha'.$$

You will be given a (surprisingly nontrivial) problem to prove that this requires

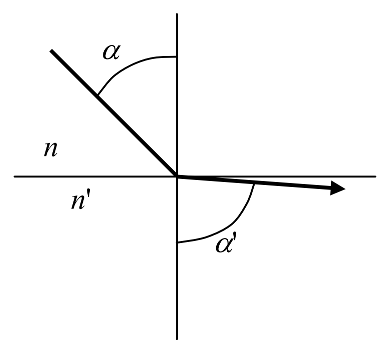
$$\alpha_B = \tan^{-1}\left(\frac{n'}{n}\right) \tag{9.1}$$

Note that $r_{\|}$ changes sign as it goes through the Brewster angle, so the direction of $\mathbf{E}_{0\|}''$ reverses (as also must $\mathbf{B}_{0\|}''$, since (7.12) requires $\mathbf{k}\times\mathbf{E}_{0\|} = \omega\mathbf{B}_{0\|}$ for any plane wave propagating in direction $\mathbf{k}$).

Many common dielectric materials (car paint, shiny road surfaces, glass) have $\frac{n'}{n_{air}} \simeq 1.5$, giving $\alpha_B \simeq 50^0$ to 60^0. Initially unpolarised sunlight becomes strongly polarised $\perp$ to the plane of incidence when it is scattered close to the Brewster angle. If we wear Polaroid sunglasses we can selectively suppress this component and reduce the glare.

9.3 Total Internal Reflection

Coming from a dense into a less dense medium ($n > n'$) then Snell's law (8.6) tells us that $\sin\alpha' = \frac{n}{n'}\sin\alpha$ so there will be a value of $\alpha \equiv \alpha_C$, the critical angle, above which $\sin\alpha' > 1$; an unphysical value.



At the critical angle $\sin\alpha' = 1$ so

$$\alpha_C = \sin^{-1}\left(\frac{n'}{n}\right) \tag{9.2}$$

For $\alpha < \alpha_C$ the Fresnel coefficients have their normal meaning. For $\alpha > \alpha_C$ we work with the unphysical angle α' as if it still obeyed the usual trigonometrical relations,

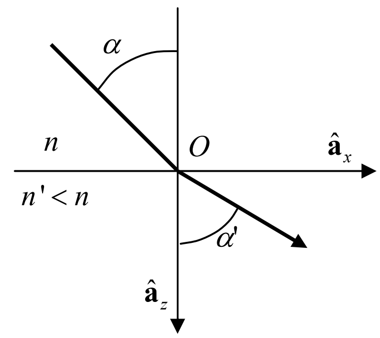
$$\cos\alpha' = \sqrt{1-\sin^2\alpha'} = \sqrt{1-\left(\frac{n}{n'}\right)^2\sin^2\alpha}\ .$$

The quantity inside the square root in this expression is equal to zero when $\alpha = \alpha_C$. When $\alpha > \alpha_C$ the quantity inside the square root becomes negative, so the root is imaginary and we can set $\cos\alpha' = iS$, where S is real and positive.

Returning to the Fresnel relations with this value of $\cos\alpha'$ we find that the reflection coefficients from (8.15) and (8.19) both have the form $r = \frac{a-ib}{a+ib} = \frac{\rho e^{-i\varphi}}{\rho e^{i\varphi}} = e^{-2i\varphi}$ with

$| r |^2 = \left(\frac{E_0 "}{E_0} \right)^2 = \frac{a^2 + b^2}{a^2 + b^2} = 1$. So all of the power is reflected, but the ratios of $\frac{E_0 "}{E_0}$ have become complex – i.e. there is a phase shift between the incident and the reflected waves.

Putting an origin in the surface, with *x*-axis in the plane of incidence and *z*-axis down into the medium *n*'. We define the angles as if everything were real, as usual, and let the mathematics tell us what is happening.



In the propagation factor $\exp i(\mathbf{k}'.\mathbf{r} - \omega t)$ for the wave below the surface
$\mathbf{k}'.\mathbf{r} = (k' \sin\alpha')x + (k' \cos\alpha')z$

As we saw above from Snell's law, $\sin\alpha' = \frac{n}{n'}\sin\alpha$ is real.

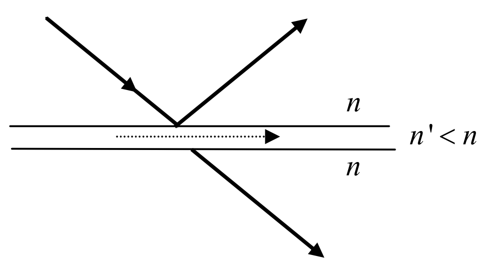
But if $\alpha > \alpha_C$ then $\cos\alpha' = iS$ is imaginary.

Putting these into the normal plane wave expression we get

$$\mathbf{E}'(\mathbf{r},t) = \mathbf{E}_0 ' \exp(-k'Sz) \exp i\{(k'\sin\alpha')x - \omega t\} \tag{9.3}$$

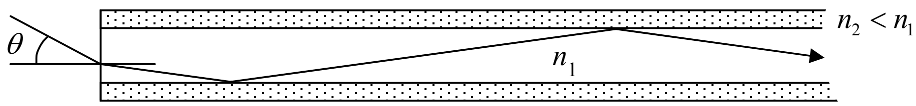
The first exponential in (9.3) has a real negative argument, so it damps the amplitude as *z* increases. The second exponential represents a travelling wave in the +*x* direction. We can think of this *evanescent wave* below the surface as a way of storing a small amount of energy in the lower-density medium. Once it is established during some very brief turn-on period, according to the requirements of the boundary conditions, then the incoming wave is totally internally reflected.

If a second piece of dense material is brought up from below to a spacing δz, close enough that the evanescent wave has a finite amplitude $\mathbf{E}_0 ' \exp(-k'S\delta z)$, then there can be another real wave in the new medium below – driven by the evanescent wave (dotted arrow in diagram below). This effect is called *frustrated total internal reflection.* It is closely analogous to quantum tunneling through a potential barrier.



9.4 Stepped Index Optical Fibres

One of many applications of total internal reflection.



A filament of glass or plastic with a high refractive index is clad with a thin layer of lower refractive index. The absorption of the core can be very small so that light can be transmitted many kilometres. An unclad fibre in air would also totally internally reflect but dirt on the surface, or objects in contact with the fibre, might have a larger refractive index than the fibre itself; in which case light would refract out and be lost. (You can show that the largest value of θ at which light can be injected through a square-cut input surface is $\theta_{\max} = \sin^{-1}\sqrt{n_1^2 - n_2^2}$).

Bundles of such fibres are used in endoscopes, each separately carrying one pixel of the image. In particle physics we use them to carry light from scintillators to remote photomultipliers or other light detectors.

Stepped index fibres are not ideal for data communications because they accept a finite bundle of angles, so the waves of a short pulse will not all take the same time to travel the length of a fibre. Instead *graded index* fibres have been developed which refract rather than reflecting the light. Matched to appropriate solid-state lasers, they preserve the sharpness of short pulses over many kilometres.