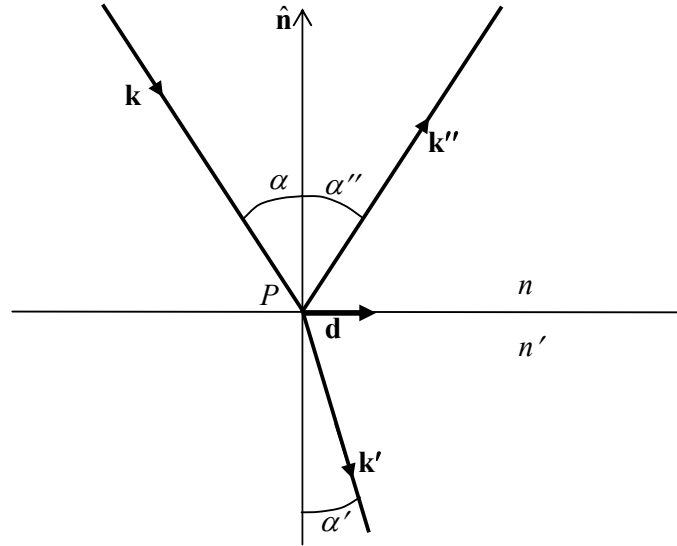


## 2B29 Electromagnetic Theory

### 8. Reflection and Refraction at a plane dielectric Surface

#### 8.1 Snell's Law and the Law of Reflection



Angle of incidence is  $\alpha$ , angle of refraction  $\alpha'$ , angle of reflection  $\alpha''$ . Upper medium has refractive index  $n$ , lower medium has  $n'$ . Both media are homogeneous and isotropic.

Take a beam of incoming plane polarised light to be an Electromagnetic plane wave with wave vector  $\mathbf{k}$ .

$$\left. \begin{aligned} \mathbf{E}(\mathbf{r}, t) &= \mathbf{E}_0 \exp i(\mathbf{k} \cdot \mathbf{r} - \omega t). \\ \text{Assume the outgoing refracted and reflected waves have the same frequency} \\ \mathbf{E}'(\mathbf{r}, t) &= \mathbf{E}_0' \exp i(\mathbf{k}' \cdot \mathbf{r} - \omega t) \\ \mathbf{E}''(\mathbf{r}, t) &= \mathbf{E}_0'' \exp i(\mathbf{k}'' \cdot \mathbf{r} - \omega t). \end{aligned} \right\} \quad (8.1)$$

[For convenience, we have allowed an arbitrary phase ( $\phi, \phi', \phi''$ ) to be absorbed into each of the constant amplitudes; i.e.  $\mathbf{E}_0 = \hat{\mathbf{e}}_0 |E_0| \exp i\phi$ , where  $\hat{\mathbf{e}}$  is a unit vector in the direction of  $\mathbf{E}_0$  and  $|E_0|$  is the modulus of the complex number  $E_0$ ; and similarly for  $\mathbf{E}_0'$  and  $\mathbf{E}_0''$ .]

Let a ray intersect the surface between the media at point  $P$  at  $\mathbf{r}_p$ . The vector  $\mathbf{d}$  represents a step away from  $P$  within the surface. The waves at the surface in this new position will have vectors

$$\left. \begin{aligned} \mathbf{E}(\mathbf{r}_p + \mathbf{d}, t) &= \mathbf{E}(\mathbf{r}_p, t) \exp i(\mathbf{k} \cdot \mathbf{d}) \\ \mathbf{E}'(\mathbf{r}_p + \mathbf{d}, t) &= \mathbf{E}'(\mathbf{r}_p, t) \exp i(\mathbf{k}' \cdot \mathbf{d}) \\ \mathbf{E}''(\mathbf{r}_p + \mathbf{d}, t) &= \mathbf{E}''(\mathbf{r}_p, t) \exp i(\mathbf{k}'' \cdot \mathbf{d}) \end{aligned} \right\} \quad (8.2)$$

that is, each wave gets a phase-shift with respect to point  $P$  when we move by  $\mathbf{d}$ .

In order for energy to be conserved and the laws of electromagnetism obeyed, there must be boundary conditions (like those in sections 2.6 and 2.7 above) at every point  $\mathbf{r}_p + \mathbf{d}$  in the surface, relating the values of  $\mathbf{E}(\mathbf{r}_p + \mathbf{d}, t)$ ,  $\mathbf{E}'(\mathbf{r}_p + \mathbf{d}, t)$  and  $\mathbf{E}''(\mathbf{r}_p + \mathbf{d}, t)$ . We do not need to know these boundary conditions in order to get the first important results, though their exact form will be needed later. For the moment let us assume only that there is a linear relationship between the three vectors, e.g. sums or differences of their components are equal to one another at all  $\mathbf{d}$ . If such a boundary condition is satisfied at  $\mathbf{r}_p$  then it will also be satisfied at any  $\mathbf{d}$  so long as the three phase shifts in (8.2) are all the same, i.e.

$$\mathbf{k} \cdot \mathbf{d} = \mathbf{k}' \cdot \mathbf{d} = \mathbf{k}'' \cdot \mathbf{d} \quad (8.3)$$

First let  $\mathbf{d}_1$  be perpendicular to  $\mathbf{k}$  (i.e. invisible in the diagram because pointing straight at us). By definition,  $\mathbf{k}$  is in the plane of the drawing, so  $\mathbf{k} \cdot \mathbf{d}_1 = 0$ , but then from (8.3)  $\mathbf{k}' \cdot \mathbf{d}_1 = \mathbf{k}'' \cdot \mathbf{d}_1 = 0$  also, so all three vectors  $\mathbf{k}$ ,  $\mathbf{k}'$  and  $\mathbf{k}''$  are coplanar. (this need not be true in general – e.g. for anisotropic crystalline materials like quartz).

Now let  $\mathbf{d}_2$  lie in the plane of  $\mathbf{k}$ ,  $\mathbf{k}'$  and  $\mathbf{k}''$ , and take  $k = k'' = 2\pi / \lambda$ , since the incident and reflected rays are both in the same medium where the wavelength is  $\lambda$ .

Then

$$\mathbf{k} \cdot \mathbf{d}_2 = \mathbf{k}' \cdot \mathbf{d}_2 = \mathbf{k}'' \cdot \mathbf{d}_2$$

$$\text{Or} \quad k \sin \alpha = k' \sin \alpha' = k'' \sin \alpha'' (= k \sin \alpha'') \quad (8.4)$$

This gives us two of the basic laws of optics:

$$\text{a)} \quad \alpha = \alpha''; \quad (8.5)$$

angle of incidence equals angle of reflection.

$$\text{b)} \quad \sin \alpha' = \frac{k}{k'} \sin \alpha$$

From (7.16)  $v_p = \frac{c}{n} = \frac{\omega}{k_n}$  for a medium with refractive index  $n$ , or  $k_n = \frac{n\omega}{c}$ .

$$\text{So } \frac{k}{k'} = \frac{n}{n'} \text{ and} \quad n \sin \alpha = n' \sin \alpha'. \quad (8.6)$$

Snell's Law.

## 8.2 Using the Boundary Conditions to get Fresnel Relations

The two laws (8.5) and (8.6) are geometrical. They tell us where the reflected and refracted waves go in the usual ideally simplified case of perfect plane waves with infinite extent in every lateral direction. (Of course, finite apertures lead to extra diffraction effects – which we do not treat here.)

But we would also like to know how much of the energy from the incoming wave is refracted into the material, and how much of it is reflected. To do this we *now* need explicit boundary conditions which relate the oscillating electric and magnetic vectors above and below the surface.

The answers are different for plane polarised waves with their electric vector  $\mathbf{E} = \mathbf{E}_{\parallel}$  contained in the plane of the wave vectors  $\mathbf{k}$ ,  $\mathbf{k}'$  and  $\mathbf{k}''$ , and waves with  $\mathbf{E} = \mathbf{E}_{\perp}$  perpendicular to the plane of  $\mathbf{k}$ ,  $\mathbf{k}'$  and  $\mathbf{k}''$ . We will use the boundary conditions on  $\mathbf{E}$ ,  $\mathbf{B}$ ,  $\mathbf{H}$  and  $\mathbf{D}$  which were derived in sections 2.6 and 2.7 above, so we need to be able to write down the wave expression for the other three vector fields as well as  $\mathbf{E}$ .

If, for instance from (8.1),  $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \exp i(\mathbf{k} \cdot \mathbf{r} - \omega t)$   
then from (1.26)  $\mathbf{D} = \varepsilon \mathbf{E}$  (8.7)  
and  $\mathbf{D}$  is always parallel to  $\mathbf{E}$ .

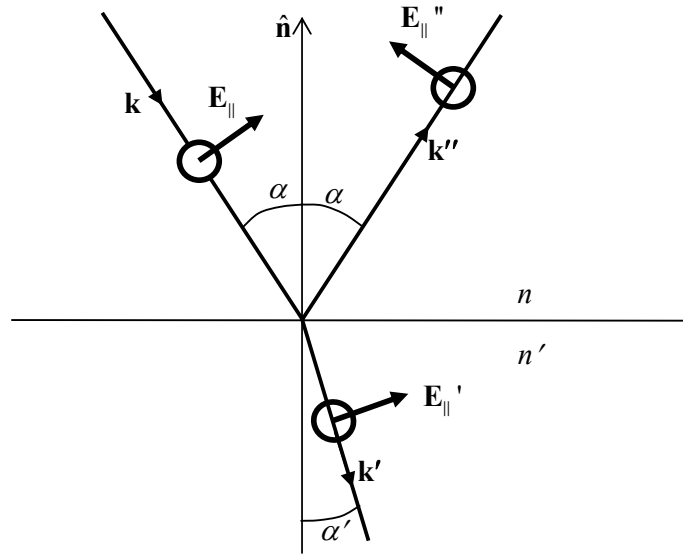
But the associated  $\mathbf{B}$  and  $\mathbf{H}$  fields will be perpendicular to the plane of the wave vectors because of (7.12)  $\mathbf{k} \times \mathbf{E}_0 = +\omega \mathbf{B}_0$ , and the equations that followed it.

Thus  $\mathbf{B}(\mathbf{r}, t) = \sqrt{\mu \varepsilon} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t)$  (8.8)

here  $\hat{\mathbf{k}}$  is the unit vector along the direction of the wavevector  $\mathbf{k}$ . Similarly, from

(2.16)  $\mathbf{H}(\mathbf{r}, t) = \frac{\mathbf{B}(\mathbf{r}, t)}{\mu} = \sqrt{\frac{\varepsilon}{\mu}} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t)$  (8.9)

When  $\mathbf{E} = \mathbf{E}_{\parallel}$  in the plane of the wavevectors, then  $\mathbf{B}$  and  $\mathbf{H}$  are out of the plane; represented on the diagram as a circle. (In this case the right-hand rule says that the magnetic vectors are out of the paper, towards us).



(When  $\mathbf{E} = \mathbf{E}_{\perp}$ , then  $\mathbf{B}$  and  $\mathbf{H}$  are in the plane of the wavevectors and the paper.)

From the boundary conditions of section 2.6 we know that, for an interface without surface charges or currents,

components of  $\mathbf{B}$  and  $\mathbf{D}$  perpendicular to the surface are conserved.

And from 2.7 we know that

components of  $\mathbf{E}$  and  $\mathbf{H}$  parallel to the surface are conserved.

We said above that if equation (8.3) ( $\mathbf{k} \cdot \mathbf{d} = \mathbf{k}' \cdot \mathbf{d} = \mathbf{k}'' \cdot \mathbf{d}$ ) holds there is no relative phase change between the  $\exp i(\mathbf{k} \cdot \mathbf{r} - \omega t)$  propagation factors of the three waves  $\mathbf{E}(\mathbf{r}_p + \mathbf{d}, t)$ ,  $\mathbf{E}'(\mathbf{r}_p + \mathbf{d}, t)$  and  $\mathbf{E}''(\mathbf{r}_p + \mathbf{d}, t)$  as we move around the surface. So we can cancel the propagation factor out and apply the boundary conditions to the vector amplitudes  $E_{0\parallel}$  etc.

For instance, taking components of  $E_{0\parallel}$  projected parallel to the surface

$$E_{0\parallel} \cos \alpha - E_{0\parallel}'' \cos \alpha = E_{0\parallel}' \cos \alpha' \quad (8.10)$$

where the left hand side represents the sum of the two relevant components of  $\mathbf{E}$  above the surface and the right hand side is the component below the surface.

The corresponding  $\mathbf{H}$  fields are given by (8.9).  $\mathbf{H}$  is  $\perp$  to  $\mathbf{k}$  so it lies entirely in the interface between the media, so continuity of the  $\parallel$  components of  $\mathbf{H}$  requires

$$\sqrt{\frac{\varepsilon}{\mu}} E_{0\parallel} + \sqrt{\frac{\varepsilon}{\mu}} E_{0\parallel}'' = \sqrt{\frac{\varepsilon'}{\mu'}} E_{0\parallel}' \quad (8.11)$$

Now we make an approximation which is reasonable for most dielectrics that

$\mu = \mu_r \mu_0 \approx \mu_0$ ; i.e.  $\mu_r \approx 1$ , and in (8.11) we can cancel  $\sqrt{\mu} \approx \sqrt{\mu'}$  from the bottom line, giving  $\sqrt{\varepsilon} E_{0\parallel} + \sqrt{\varepsilon} E_{0\parallel}'' = \sqrt{\varepsilon'} E_{0\parallel}'$ . Cancelling  $\varepsilon_0$ ,  $\sqrt{\varepsilon_r} E_{0\parallel} + \sqrt{\varepsilon_r} E_{0\parallel}'' = \sqrt{\varepsilon_r'} E_{0\parallel}'$ .

Then from equation (7.16) and nearby arguments

$$n = \frac{c}{v_p} = \sqrt{\frac{\mu \varepsilon}{\mu_0 \varepsilon_0}} \approx \sqrt{\frac{\varepsilon}{\varepsilon_0}} = \sqrt{\varepsilon_r} \quad (8.12)$$

So (8.11) becomes

$$n E_{0\parallel} + n E_{0\parallel}'' = n' E_{0\parallel}' \quad (8.13)$$

To get a simple relation between the incoming and the reflected amplitudes we eliminate  $E_{0\parallel}'$  by dividing equation (8.10) by (8.13).

$$\frac{\cos \alpha}{n} \left( \frac{E_{0\parallel} - E_{0\parallel}''}{E_{0\parallel} + E_{0\parallel}''} \right) = \frac{\cos \alpha'}{n'}$$

Define

$$r_{\parallel} \equiv \frac{E_{0\parallel}''}{E_{0\parallel}} \quad (8.14)$$

to be the *reflection coefficient* for a wave with  $\mathbf{E}(\mathbf{r}, t)$  parallel to the plane of the wave vectors. Then

$$n \cos \alpha' = n' \cos \alpha \left( \frac{1 - r_{\parallel}}{1 + r_{\parallel}} \right)$$

so

$$r_{\parallel} = \frac{n' \cos \alpha - n \cos \alpha'}{n' \cos \alpha + n \cos \alpha'}$$

or

$$r_{\parallel} = \frac{\frac{n'}{n} \cos \alpha - \cos \alpha'}{\frac{n'}{n} \cos \alpha + \cos \alpha'} \quad (8.15)$$

Now define the *transmission coefficient* for a wave with  $\mathbf{E}(\mathbf{r},t)$  parallel to the plane of the wave vectors

$$t_{\parallel} \equiv \frac{E_{0\parallel}'}{E_{0\parallel}}. \quad (8.16)$$

Eliminating  $E_{0\parallel}''$  between rearranged versions of (8.10) and (8.13)

$$E_{0\parallel} \cos \alpha - E_{0\parallel}' \cos \alpha' = E_{0\parallel}'' \cos \alpha$$

and

$$\begin{aligned} nE_{0\parallel} - n'E_{0\parallel}' &= -nE_{0\parallel}'' \\ \frac{-\cos \alpha}{n} &= \frac{E_{0\parallel} \cos \alpha - E_{0\parallel}' \cos \alpha'}{nE_{0\parallel} - n'E_{0\parallel}'} \end{aligned}$$

Then

$$\cos \alpha = \frac{t_{\parallel} \cos \alpha' - \cos \alpha}{1 - \frac{n'}{n} t_{\parallel}}$$

or

$$\cos \alpha - \left(\frac{n'}{n}\right) t_{\parallel} \cos \alpha = t_{\parallel} \cos \alpha' - \cos \alpha$$

so

$$t_{\parallel} = \frac{2 \cos \alpha}{\left(\frac{n'}{n}\right) \cos \alpha + \cos \alpha'} \quad (8.17)$$

We can use very similar arguments for waves with the other polarisation  $\mathbf{E} = \mathbf{E}_{\perp}$  (In this case we actually use the continuity of  $\mathbf{D}$  and  $\mathbf{B}$  perpendicular to the interface – do it for yourself, or look in textbooks.) This gives a reflection coefficient

$$r_{\perp} \equiv \frac{E_{0\perp}''}{E_{0\perp}} \quad (8.18)$$

$$= \frac{\cos \alpha - \frac{n'}{n} \cos \alpha'}{\cos \alpha + \frac{n'}{n} \cos \alpha'} \quad (8.19)$$

and a transmission coefficient

$$t_{\perp} \equiv \frac{E_{0\perp}'}{E_{0\perp}} \quad (8.20)$$

$$t_{\perp} = \frac{2 \cos \alpha}{\cos \alpha + \frac{n'}{n} \cos \alpha'} \quad (8.21)$$

The four equations (8.15), (8.17), (8.19) and (8.21) are called the *Fresnel relations*. (You may be asked to use or manipulate them in examination, but will not be asked for a full derivation)