2B29 Electromagnetic Theory

7. More about Electromagnetic Waves

7.1 Waves in E and D

Now let us go down a closely similar path for waves in **E**(**r**,t) to the path we took for **H**(**r**,t). Start by taking the curl of (6.2)

$$\nabla\times\nabla\times\mathbf{E} = -\frac{\partial}{\partial t}\nabla\times\mathbf{B}$$

with $$\nabla\times\mathbf{B} = \mu\nabla\times\mathbf{H} = \mu\left(\mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}\right) \text{ from (6.4)}$$

and in the general case of a conductive medium, as before, we can use (6.12) $\mathbf{J} = \sigma\mathbf{E}$ and (6.9) $\mathbf{D} = \varepsilon\mathbf{E}$ to get $$\nabla\times\mathbf{B} = \mu\sigma\mathbf{E} + \mu\varepsilon\frac{\partial\mathbf{E}}{\partial t}.$$

Our Tools give $$\nabla\times\nabla\times\mathbf{E} = \nabla.(\nabla.\mathbf{E} - \nabla^2\mathbf{E})$$

There is something new here that did not crop up in the equivalent part of the derivation of (6.14) because (6.1) says $\nabla.\mathbf{D} = \varepsilon\nabla.\mathbf{E} = \rho_f$, so $\nabla.\mathbf{E}$ is not necessarily zero (whereas $\nabla.\mathbf{B} = 0$ because there are no monopoles). In fact we do nothing in this course which requires a wave equation for systems with concentrations of free charge so we do not investigate this more general case. It would apply, for instance, if we were studying waves in the particle bunches in an accelerator. When we talk later about waves in plasmas or waves in metals they are always associated with the movement of charges inside bulk neutral material. On this basis we set

$$\nabla.\mathbf{E} = \rho_f = 0 \tag{7.1}$$

and get

$$\nabla^2\mathbf{E} - \sigma\mu\frac{\partial\mathbf{E}}{\partial t} - \varepsilon\mu\frac{\partial^2\mathbf{E}}{\partial t^2} = 0 \tag{7.2}$$

so **E**(**r**,t) obeys the identical equation to (6.14) for **H**(**r**,t). If we turn off the conductivity then we get the nondispersive wave equation again

$$\nabla^2\mathbf{E} = \varepsilon\mu\frac{\partial^2\mathbf{E}}{\partial t^2} \tag{7.3}$$

If we multiply both sides by ε and use $\mathbf{D} = \varepsilon\mathbf{E}$ then the wave equation for D is just the same

$$\nabla^2\mathbf{D} = \varepsilon\mu\frac{\partial^2\mathbf{D}}{\partial t^2}. \tag{7.4}$$

7.2 Linking the Electric and Magnetic parts of the Waves.

Plane wave solutions to (6.15) for **H**, to (6.19) for **B**, to (7.3) for **E** and to (7.4) for **D** all take the same form, with a real constant relating **H** and **B**, or **E** and **D**. But what is the relation between **E**(**r**,t) and **B**(**r**,t)? To find out let us write them both down in the standard plane wave form, c.f. (6.16):

$$\mathbf{B}(\mathbf{r},t)=\mathbf{B}_0 \exp i\left(\mathbf{k}_B.\mathbf{r}-\omega_B t+\phi_B\right) \tag{7.5}$$

and

$$\mathbf{E}(\mathbf{r},t)=\mathbf{E}_0 \exp i\left(\mathbf{k}_E.\mathbf{r}-\omega_E t+\phi_E\right) \tag{7.6}$$

(Note; the order of the $(\mathbf{k.r}-\omega t)$ bit in any wave does not matter (see problem) but the relative negative sign is important.)

We can relate (7.5) and (7.6) by using the obvious Maxwell equation:

the Faraday law (6.2) $\qquad \nabla\times\mathbf{E}=-\dfrac{\partial\mathbf{B}}{\partial t}$.

For any plane wave, e.g.

$$\mathbf{C}(\mathbf{r},t)=\mathbf{C}_0 \exp i\left(\mathbf{k.r}-\omega t+\phi\right), \tag{7.7}$$

we can use *three important manipulation formulae* which come from the laws of differentiation:

$$\frac{\partial\mathbf{C}(\mathbf{r},t)}{\partial t}=-i\omega\mathbf{C}(\mathbf{r},t) \tag{7.8}$$

$$\nabla.\mathbf{C}(\mathbf{r},t)=i\mathbf{k}.\mathbf{C}(\mathbf{r},t) \tag{7.9}$$

$$\nabla\times\mathbf{C}(\mathbf{r},t)=i\mathbf{k}\times\mathbf{C}(\mathbf{r},t) \tag{7.10}$$

(7.8) is reasonably obvious. It is easy to check the other two in Cartesian co-ordinates with $\nabla=\left(\hat{\mathbf{i}}\dfrac{\partial}{\partial x}+\hat{\mathbf{j}}\dfrac{\partial}{\partial y}+\hat{\mathbf{k}}\dfrac{\partial}{\partial z}\right)$ and $\mathbf{k.r}=\left(k_x x+k_y y+k_z z\right)$.

E.g. for (7.9) $\nabla.\mathbf{C}=\left(\hat{\mathbf{i}}ik_x+\hat{\mathbf{j}}ik_y+\hat{\mathbf{k}}ikz\right).\mathbf{C}_0 \exp i\left(\mathbf{k.r}-\omega t+\phi\right)=i\mathbf{k}.\mathbf{C}(\mathbf{r},t)$.

There is a similar argument for (7.10).

Using (7.10) and (7.8) in the Faraday law we get

$$i\mathbf{k}_E\times\mathbf{E}=+i\omega_B\mathbf{B},$$

or, writing this out in full,

$$\mathbf{k}_E\times\mathbf{E}_0 \exp i\left(\mathbf{k}_E.\mathbf{r}-\omega_E t+\phi_E\right)=+\omega_B\mathbf{B}_0 \exp i\left(\mathbf{k}_B.\mathbf{r}-\omega_B t+\phi_B\right) \tag{7.11}$$

The time t and the three components of **r** are all independent variables. So if (7.11) is to be true at all points in spacetime then **E**(**r**,t) and **B**(**r**,t) are inextricably linked into a single disturbance with both electrical and magnetic aspects – an electromagnetic wave – in which

$$\omega_B=\omega_E=\omega,$$

$$\mathbf{k}_B=\mathbf{k}_E=\mathbf{k},$$

and

$$\phi_B=\phi_E=\phi.$$

(We often choose $\phi=0$ for simplicity)

Substituting back into (7.11) we see that the "propagation factors" $\exp i\left(\mathbf{k.r}-\omega t+\phi\right)$ are now the same on both sides, so they cancel, giving

$$\mathbf{k}\times\mathbf{E}_0=+\omega\mathbf{B}_0 \tag{7.12}$$

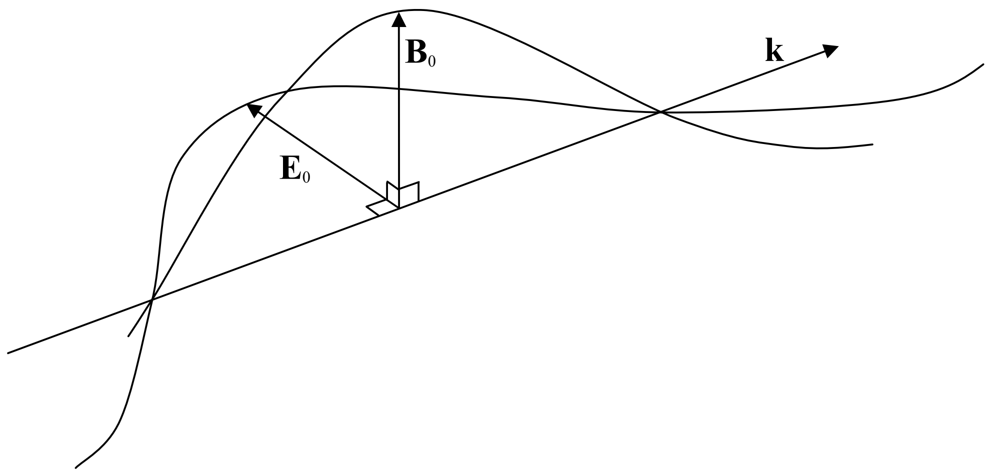
You will show in a problem that **k** is a vector along the direction of propagation of the wave.

Clearly from (7.12) $\mathbf{B}_0$ is perpendicular to both $\mathbf{k}$ and $\mathbf{E}_0$.

We decided at (7.1) to set $\nabla.\mathbf{E} = \rho_f = 0$, so from (7.9)

$\nabla.\mathbf{E} = i\mathbf{k}.\mathbf{E} = 0$, which means $\mathbf{k}.\mathbf{E}_0 = 0$, so $\mathbf{k}$ and $\mathbf{E}_0$ are perpendicular.

(In the same way, the no-monopoles law (6.3) says $\nabla.\mathbf{B} = 0$, so $\mathbf{k}$ and $\mathbf{B}_0$ are perpendicular, but we already deduced that from (7.12))



Equation (7.12) tells us that **k**, **E** and **B** form a right-handed set. If one of **E** or **B** is reversed then **k** reverses; i.e. the wave goes in the opposite direction.

Because both **E** and **B** are perpendicular to **k** this kind of wave is called "TEM", i.e. Transverse Electric and Magnetic.

Equation (7.12) can also be used to relate the sizes of the electric and magnetic components of the single plane electromagnetic wave which we are now describing. Because the three vectors are orthogonal we have

$$kE_0 = \omega B_0 . \tag{7.13}$$

But from (6.17) the phase velocity $\mathrm{v_p} = \dfrac{\omega}{k} = \dfrac{1}{\sqrt{\varepsilon\mu}}$.

So
$$B_0 = \frac{kE_0}{\omega} = \frac{E_0}{\mathrm{v_p}} = \sqrt{\varepsilon\mu}E_0 \tag{7.14}$$

$$= \frac{E_0}{\mathrm{c}} \text{ in vacuum.} \tag{7.15}$$

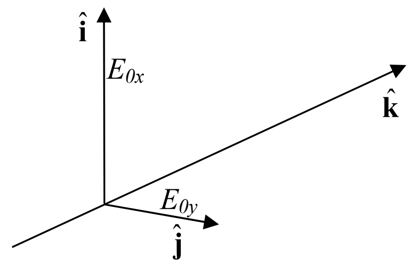
In the next section, where we look at the wave-basis of Optics, we also need (from 1B24) that the refractive index of a uniform linear medium is

$$n = \frac{c}{\mathrm{v_p}} \tag{7.16}$$

7.2 Polarisation of Electromagnetic Waves

What was described on the previous page was a *plane-polarised* wave. $\mathbf{E}_0$ and $\mathbf{B}_0$ are perpendicular to one another and to the wavevector $\mathbf{k}$ and they stay fixed in time.

But there are two spatial dimension in the plane perpendicular to $\mathbf{k}$ (which can be taken as the *x*-*y* plane, with $\mathbf{k}$ along the *z*-direction). For the same frequency and wavelength we can have two independent components of the wave with different amplitudes and phases (E_{0x}, ϕ_x) and (E_{0y}, ϕ_y);



$$\mathbf{E}_x(\mathbf{r},t) = E_{0x}\hat{\mathbf{i}}\exp i(kz-\omega t+\phi_x) \tag{7.17}$$

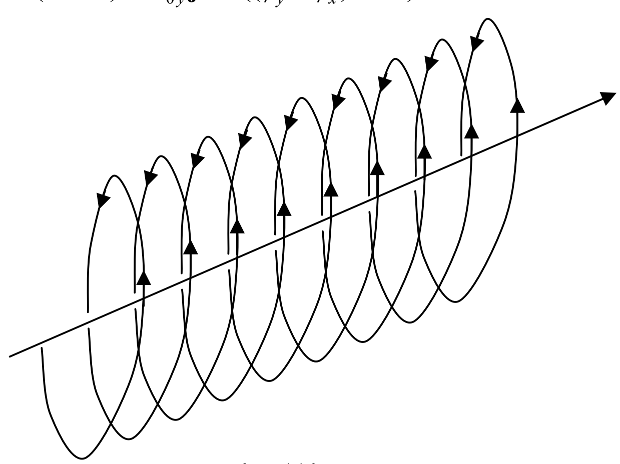
$$\mathbf{E}_y(\mathbf{r},t) = E_{0y}\hat{\mathbf{j}}\exp i(kz-\omega t+\phi_y) \tag{7.18}$$

At a given point in z the sum of these two waves can be written

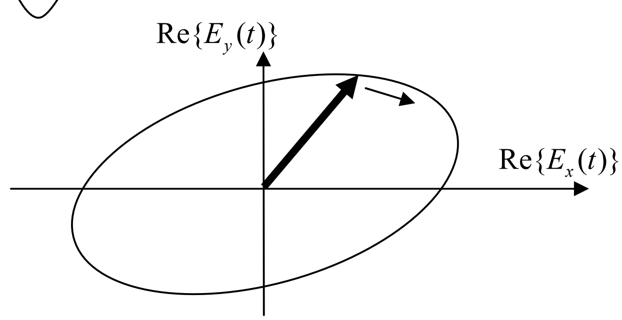
$$\mathbf{E}(\mathbf{r},t) = \exp i(kz+\phi_x)\Big(E_{0x}\hat{\mathbf{i}}\exp i(-\omega t+) + E_{0y}\hat{\mathbf{j}}\exp i((\phi_y-\phi_x)-\omega t)\Big)$$

where the first exponential factor is a constant, an overall arbitrary phase which can be factored out, to get $\mathbf{E}(t;z) = E_{0x}\hat{\mathbf{i}}\exp i(-\omega t+) + E_{0y}\hat{\mathbf{j}}\exp i((\phi_y-\phi_x)-\omega t)$. Looking only at the time-variation, and taking the Real part, since that is what has physical meaning, we get $\mathrm{Re}(\mathbf{E}(t;z)) = E_{0x}\hat{\mathbf{i}}\cos(-\omega t+) + E_{0y}\hat{\mathbf{j}}\cos((\phi_y-\phi_x)-\omega t)$

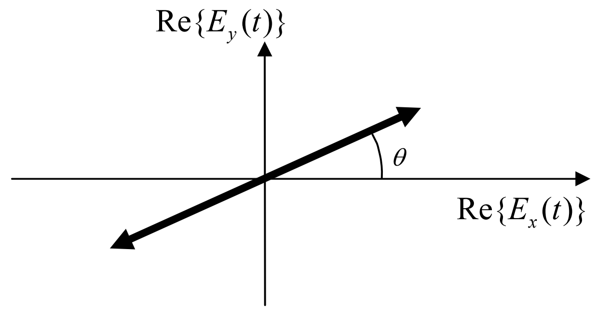
Instead of oscillating in a plane perpendicular to $\mathbf{k}$, as sketched on the previous page, the tip of the electric field vector will in general follow a spiral path around $\mathbf{k}$, a bit like a corkscrew, but the amplitudes E_{0x}, E_{0y} of the two components do not have to be equal in size {as they must in corkscrews which draw corks}.



The path of the tip of the Electric vector can best be pictured in projection onto the *x*-*y* plane where, in general, it has an *elliptical* orbit like the Lissajous figures that can be drawn on an oscilloscope by putting an AC signal onto the *x* deflection plates and a second version of the same signal with different amplitude and/or phase on the *y* deflection plates.



In the special case where $\phi_x = \phi_y$, with $E_{0x} \neq E_{0y}$ then the *plane* of polarisation is at $\theta = \tan^{-1}\left(\frac{E_{0y}}{E_{0x}}\right)$ from the *x* axis.



7.3 Classification of kinds of polarisation

a) Plane polarisation. The electric vector oscillates in a plane through the axis **k** of propagation.

b) Elliptical polarisation. The tip of the electric vector rotates in an ellipse in the plane perpendicular to **k**.

c) Circular polarisation. Can be regarded as special case of elliptical polarisation with both orthogonal components of the wave having the same amplitude $E_{0x} = E_{0y}$, but $\pi/2$ or $3\pi/2$ out of phase; a true corkscrew (can you work out which phase of $\phi_x - \phi_y$ gives a right-handed thread?)

d) Unpolarised waves are an incoherent mixture of many independent emissions each with an arbitrary phase and state of polarisation.

e) Partially polarised waves may be a mixture of unpolarised waves with one or more kinds of polarised waves; a), b) and c). We can also partially polarise unpolarised light by preferentially absorbing waves from the mixture whose plane of polarisation is in preferred direction perpendicular to **k** (as with Polaroid sunglasses – see later).

Lasers and radio transmitters tend to emit waves with a high degree of polarisation (often close to ideally plane or circular).

Unmagnetised extended light-sources tend to emit unpolarised light – e.g. a light bulb or the sun.

Magnetisation of an emitting source (especially a plasma) may give rise to partial polarisation.