2B29 Electromagnetic Theory.

iii) Some useful Mathematical Tools

VECTORS

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

VECTOR CALCULUS

$$\nabla\times\nabla\,\phi = 0$$

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0$$

$$\nabla\cdot(\phi\,\mathbf{A}) = \phi\,\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla\phi$$

$$\nabla\times(\phi\,\mathbf{A}) = \nabla\phi\times\mathbf{A} + \phi\nabla\times\mathbf{A}$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A})$$

* $$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = (\nabla\times\mathbf{A})\cdot\mathbf{B} - (\nabla\times\mathbf{B})\cdot\mathbf{A}$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$$

* $$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$

where

$$(\mathbf{A}\cdot\nabla)\mathbf{B} = \hat{\mathbf{i}}\left(A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}\right)B_x + \hat{\mathbf{j}}\left(A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}\right)B_y + \hat{\mathbf{k}}\left(A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}\right)B_z$$

For a function that depends only on the distance $r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$, e.g. as $\phi(r)$, $\mathbf{F}(r)$, then

$$\nabla \equiv \frac{\mathbf{r}}{r}\frac{\mathrm{d}}{\mathrm{d}r}, \quad \text{i.e.} \quad \nabla\phi(r) = \frac{\mathbf{r}}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r}, \quad \nabla\cdot\mathbf{F}(r) = \frac{\mathbf{r}}{r}\cdot\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}r}, \quad \nabla\times\mathbf{F}(r) = \frac{\mathbf{r}}{r}\times\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}r},$$

$$\nabla^2\phi = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) = \frac{1}{r}\frac{\mathrm{d}^2(r\,\phi)}{\mathrm{d}r^2} = \frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r}.$$

EXPLICIT FORMS OF THE VECTOR OPERATORS

(a) Cartesian (x, y, z), volume element $\mathrm{d}\tau = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.

$$\nabla\psi = \hat{\mathbf{i}}\frac{\partial\psi}{\partial x} + \hat{\mathbf{j}}\frac{\partial\psi}{\partial y} + \hat{\mathbf{k}}\frac{\partial\psi}{\partial z} \quad \text{and} \quad \nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \hat{\mathbf{i}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{j}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{k}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla\times\mathbf{A} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla\cdot\nabla\psi = \nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

(b) Cylindrical polar (ρ, ϕ, z), volume element $d\tau = \rho \, d\rho \, d\phi \, dz$.

$$\nabla \psi = \hat{\boldsymbol{\rho}} \frac{\partial \psi}{\partial \rho} + \hat{\boldsymbol{\phi}} \frac{1}{\rho} \frac{\partial \psi}{\partial \phi} + \hat{\mathbf{z}} \frac{\partial \psi}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial (\rho A_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \frac{1}{\rho} \begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

(c) Spherical polar (r, θ, ϕ), volume element $d\tau = r^2 \sin\theta \, dr \, d\theta \, d\phi$.

$$\nabla \psi = \hat{\mathbf{r}} \frac{\partial \psi}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial \psi}{\partial \theta} + \hat{\boldsymbol{\phi}} \frac{1}{r \sin\theta} \frac{\partial \psi}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial (\sin\theta \, A_\theta)}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{r \sin\theta} \begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r \sin\theta \hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & r \, A_\theta & r \sin\theta \, A_\phi \end{vmatrix}$$

$$\nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \psi}{\partial \phi^2}$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) = \frac{1}{r} \frac{\partial^2 (r\psi)}{\partial r^2} = \frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r} \frac{\partial \psi}{\partial r}$$

INTEGRAL RELATIONS

If V is a volume with volume element $d\tau$ and S is a closed surface enclosing volume V and having a surface element dS and an outward normal $\hat{\mathbf{n}}$ at dS then

* $$\int_V \nabla \cdot \mathbf{A} \, d\tau = \oint_S \mathbf{A} \cdot \hat{\mathbf{n}} \, dS \qquad \text{(Divergence theorem)}$$

If S is an open surface and C is a contour bounding it with line element $\mathbf{d}\boldsymbol{\ell}$ and the normal $\hat{\mathbf{n}}$ to S is defined by the right-hand rule in relation to the sense of a line integral around C then

* $$\int_S (\nabla \times \mathbf{A}) \cdot \hat{\mathbf{n}} \, dS = \oint_C \mathbf{A} \cdot \mathbf{d}\boldsymbol{\ell} \qquad \text{(Stokes's theorem)}$$