2B29 Electromagnetic Theory

6. Maxwell's Equations and Magnetic Plane Waves

6.1 Summary of Maxwell's Equations

Now let us tidy up and renumber Maxwell's equations for future use:

Gauss' law (1.21)

$$\nabla.\mathbf{D} = \rho_f \tag{6.1}$$

Faraday's law of induction (1.32)

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{6.2}$$

"No magnetic monopoles" (1.28)

$$\nabla.\mathbf{B} = 0 \tag{6.3}$$

Ampere's law, augmented by Maxwell, (5.5)

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \tag{6.4}$$

These are connected by Stokes' theorem or the Gauss divergence theorem to their integral forms:

Gauss' law (c.f. (1.6))

$$\oint_S \mathbf{D}.d\mathbf{S} = \int_\tau \rho_f d\tau = \sum_\tau q_i \tag{6.5}$$

Faraday's law of induction (c.f. (1.30))

$$\oint_C \mathbf{E}.d\mathbf{l} = -\int_{S_C} \frac{\partial \mathbf{B}}{\partial t}.d\mathbf{S} = -\frac{d\Phi_C}{dt} \tag{6.6}$$

"No magnetic monopoles", (1.28)

$$\oint_S \mathbf{B}.d\mathbf{S} = 0 \tag{6.7}$$

Ampere's law, augmented by Maxwell, (5.7)

$$\oint_P \mathbf{H}.d\mathbf{l} = \int_{S_P} \left(\mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \right).d\mathbf{S} \tag{6.8}$$

These equations summarise the fundamentals of everything you have done in 1B26 and in this course until now; Electricity and Magnetism. But they also provide the basis of the new integrated subject of Electromagnetism. The first step is to turn them into wave equations which govern the behaviour of electromagnetic waves. We will find that the electrical and the magnetic properties of these waves are inextricably linked at every point in space.

6.2 Plane Magnetic Waves

At first sight there seems to be the possibility of an independent wave equation for each of **E**, **B**, **D** and **H** , but if we assume a linear isotropic medium we can immediately halve the number of degrees of freedom by writing:

from (1.26) $$\mathbf{D} = \varepsilon_0 \varepsilon_r \mathbf{E} = \varepsilon \mathbf{E} \tag{6.9}$$

and from (2.16) $$\mathbf{B} = \mu_0 \mu_r \mathbf{H} = \mu \mathbf{H} \tag{6.10}$$

Since wave equations are second order, the next move is to differentiate a Maxwell equation, the Ampere law for instance (6.4)

$$\nabla \times \nabla \times \mathbf{H} = \nabla \times \left(\mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \right) \tag{6.11}$$

and for the moment let us assume a medium with finite conductivity, so $\mathbf{J}(\mathbf{r},t) \neq 0$. This means that if $\mathbf{E}(\mathbf{r},t) \neq 0$ a current must flow. If the sole cause of the current is the **E** field then we can have

$$\mathbf{J} = \sigma \mathbf{E} \tag{6.12}$$

where σ is the conductivity. Our Mathematical tools tell us that

$$\nabla \times \nabla \times \mathbf{H} = \nabla (\nabla . \mathbf{H}) - \nabla^2 \mathbf{H} \tag{6.13}$$

and $\nabla . \mathbf{H} = \dfrac{\nabla . \mathbf{B}}{\mu} = 0$ from (6.3) and (6.10).

Using (6.9) and (6.11), $-\nabla^2 \mathbf{H} = \sigma \nabla \times \mathbf{E} + \varepsilon \dfrac{\partial}{\partial t} \nabla \times \mathbf{E}$.

But from (6.10) $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} = -\mu \dfrac{\partial \mathbf{H}}{\partial t}$

so $$\nabla^2 \mathbf{H} - \sigma\mu \frac{\partial \mathbf{H}}{\partial t} - \varepsilon\mu \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \tag{6.14}$$

This is a wave equation for $\mathbf{H}(r,t)$ with a damping term proportional to $\sigma\mu$ - to be expected because finite resistance dissipates energy (important in plasmas, metals etc. etc. – some of which are treated later). Notice that the second time derivative $\dfrac{\partial^2}{\partial t^2}$ comes from both the displacement current term $\dfrac{\partial \mathbf{D}}{\partial t}$ in (6.4) and the $\dfrac{\partial \mathbf{B}}{\partial t}$ term in (6.2). We cannot have the wave equation without the displacement current.

For the present, let $\sigma \to 0$, so we are now in an homogeneous, isotropic nonconducting medium; e.g. the vacuum, a piece of glass etc. etc.

Then $$\nabla^2 \mathbf{H} = \varepsilon\mu \frac{\partial^2 \mathbf{H}}{\partial t^2}. \tag{6.15}$$

This is the standard "nondispersive" wave equation – though paradoxically the solutions of it may have dispersion; we'll see how later. [Check that you remember what dispersion is!]

Let's try a solution representing a plane wave in 3-dimensions

$$\mathbf{H}(\mathbf{r},t) = \mathbf{H}_0 \exp i (\mathbf{k}.\mathbf{r} - \omega t + \phi) \tag{6.16}$$

[There will be a problem to make you more familiar with this expression. You need to understand it very thoroughly.] {This course will use either $i=\sqrt{-1}$ or $j = \sqrt{-1}$ interchangeably . Engineers and physicists never agree which is correct.}

To show that (6.16) is a solution to (6.15) we can choose to work in Cartesian co-ordinates for a moment. Then

$$\nabla^2\mathbf{H} = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\mathbf{H}_0 \exp i\left(\{\hat{\mathbf{i}}k_x+\hat{\mathbf{j}}k_y+\hat{\mathbf{k}}k_z\}.\{\hat{\mathbf{i}}x+\hat{\mathbf{j}}y+\hat{\mathbf{k}}z\}-\omega t+\phi\right)$$

$$= -\{k_x^2+k_y^2+k_z^2\}\mathbf{H}_0 \exp i\left(\mathbf{k.r}-\omega t+\phi\right) = -k^2\mathbf{H}\,.$$

And
$$\frac{\partial^2\mathbf{H}}{\partial t^2} = -\omega^2\mathbf{H}\,.$$

Therefore if (6.16) is to satisfy (6.15)

$$k^2\mathbf{H} = \varepsilon\mu\omega^2\mathbf{H}$$

so
$$\frac{\omega^2}{k^2} = \frac{1}{\varepsilon\mu}\,.$$

You will remember from 1B24 that the phase velocity of a wave $\mathrm{v_p} = \frac{\omega}{k}$, where ω is the angular frequency $\{= 2\pi\nu\}$ and k is the wavenumber $\{= \frac{2\pi}{\lambda}\}$.

So
$$\mathrm{v_p} = \frac{1}{\sqrt{\varepsilon\mu}} \tag{6.17}$$

Note. Although we say that (6.15) is nondispersive, if ε_r is a function of ω then $\mathrm{v_p}$ will vary with frequency so there will be dispersion. The wave is only truly nondispersive if $\varepsilon\mu$ is constant, which applies strictly to only one medium; the vacuum, where $\varepsilon = \varepsilon_r\varepsilon_0 \to \varepsilon_0$ and $\mu = \mu_r\mu_0 \to \mu_0$. In the vacuum

so
$$\nabla^2\mathbf{H} = \varepsilon_0\mu_0\frac{\partial^2\mathbf{H}}{\partial t^2} \quad \text{and} \quad \mathrm{c} = \frac{1}{\sqrt{\varepsilon_0\mu_0}}\,. \tag{6.18}$$

[**Nonexaminable.**
This was Einstein's starting point for Special Relativity; from a

postulate "the laws of physics are the same in all inertial frames".
Maxwell's equations are part of the laws of physics, so measurements of ε_0 and μ_0 are the same in all inertial frames. Hence, from (6.18) comes his

corollary "the speed of light is the same in all inertial frames".
He used the corollary to derive the Lorentz transformations and all the other results of Special Relativity, including $E = mc^2$. **End nonexaminable.**]

Equation (6.15) and the linearity assumption $\mathbf{B} = \mu\mathbf{H}$ (6.10) give us the wave equation for $\mathbf{B}$
$$\nabla^2\mathbf{B} = \varepsilon\mu\frac{\partial^2\mathbf{B}}{\partial t^2} \tag{6.19}$$