Answer all SIX questions in SECTION A and THREE from SECTION B.
The numbers in square brackets at the right-hand side of the text indicate the provisional allocation of maximum marks per question or sub-section of a question.

You may need:

Permeability of free space $4\pi \times 10^{-7}$ H m^{-1}.
Permittivity of free space 8.854×10^{-12} F m^{-1}.
Speed of light *in vacuo* 3×10^{8} m s^{-1}.
Charge on electron 1.6022×10^{-19} C
Mass of electron 9.1094×10^{-31} kg
For any vector field **A:**

$\nabla . \nabla \times \mathbf{A} = 0$

$\nabla \times \nabla \times \mathbf{A} = \nabla\left(\nabla . \mathbf{A}\right) - \nabla^2 \mathbf{A}$.

the Stokes theorem is $\oint_C \mathbf{A}.d\mathbf{l} = \int_S (\nabla \times \mathbf{A}).d\mathbf{s}$

the Gauss divergence theorem is $\oint_S \mathbf{A}.d\mathbf{s} = \int_V (\nabla . \mathbf{A}) d\mathrm{v}$

Phase velocity of a plane wave $\mathrm{v_p} = \dfrac{\omega}{k}$. Group velocity $\mathrm{v_g} = \dfrac{\partial \omega}{\partial k}$.

$\sqrt{i} = \dfrac{1}{\sqrt{2}}\left(1 + i\right)$

SECTION A

1. The Faraday law of electromagnetic induction for a circuit C can be written in the form

$$\oint_C \mathbf{E}.d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t}.d\mathbf{s}$$

Define all the symbols in this expression and explain what it means. [3]

Use the Stokes theorem to transform it into the Maxwell equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

giving a brief justification for each step. [3]

PLEASE TURN OVER

2. The Gauss law for electricity can be written in the form

$$\oint_S \mathbf{E}.d\mathbf{s} = \frac{1}{\varepsilon_0}\int_V \rho d\mathrm{v}\,.$$

Define all the symbols in this expression and explain what it means. [3]

Use the Gauss divergence theorem to transform it into the equation $\nabla.\mathbf{E} = \rho/\varepsilon_0$ giving a brief justification for each step. [3]

3. Show, starting with the appropriate Maxwell equation, that normal components of **B** are continuous across a plane boundary between media with different magnetic properties. [5]

What does this imply for the conservation of lines of **B?** [2]

4. Show that, for a plane wave of the form $\mathbf{E} = \mathbf{E}_0 \exp i(\omega t - \mathbf{k.r})$, $\nabla.\mathbf{E} = -i\mathbf{k.E}$. [3]
Use this result, together with the analogous relations $\nabla \times \mathbf{E} = -i\mathbf{k}\times\mathbf{E}$ and $\dfrac{\partial \mathbf{E}}{\partial t} = i\omega\mathbf{E}$, to show from the Maxwell equations that the directions of the **E** and **H** vectors and the propagation vector of an unbounded plane wave in free space are mutually perpendicular. [4]

5. The dispersion relation for electromagnetic waves in a plasma is

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right).$$

Briefly define the quantities in this equation. [2]

Use this relation to show that if $\omega > \omega_p$ one of either the group velocity or the phase velocity must be greater than the speed of light in vacuum. State briefly how this can be made compatible with the predictions of the special theory of relativity. [5]

6. What is meant by TEM waves? Explain, with the help of a simple diagram, why they can not be transmitted for useful distances along a waveguide with rectangular cross section? [3]

TM_{lm} and TE_{lm} waves can be transmitted along such a guide. Explain what TE and TM mean. and describe, with a sketch, the role played by the mode numbers l and m. [4]

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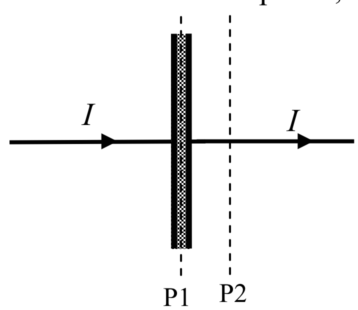
SECTION B

7. Explain the difference between the concepts of polarisation charge density ρ_p and free charge density $\rho_{f.}$ How are they related to the total charge density ρ? The polarization charge density in a medium can be obtained from the expression $\nabla.\mathbf{P} = -\rho_p$. What is the name of the vector field **P** and what are its units? Defining the displacement $\mathbf{D}(\mathbf{r}) \equiv (\varepsilon_0 \mathrm{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r}))$, show that $\nabla.\mathbf{D} = \rho_f$. [4]

Define the dielectric constant κ in terms of the behaviour of a parallel plate capacitor when the gap between the plates is initially empty and is then filled with dielectric material. How is the dielectric constant related to the permittivity of the medium between the plates? Use an integral form of the Gauss law to show how the magnitude of **D** inside the capacitor varies with the charge on the plates. How does it vary with κ? [6]

Give an expression for the displacement current density in terms of the value of **D** at a point. What is its value inside a capacitor with circular plates of radius 5 mm and an empty gap of width d = 0.01 mm when a current of 1A is flowing in through straight wires connected to the centres of the plates, as shown?

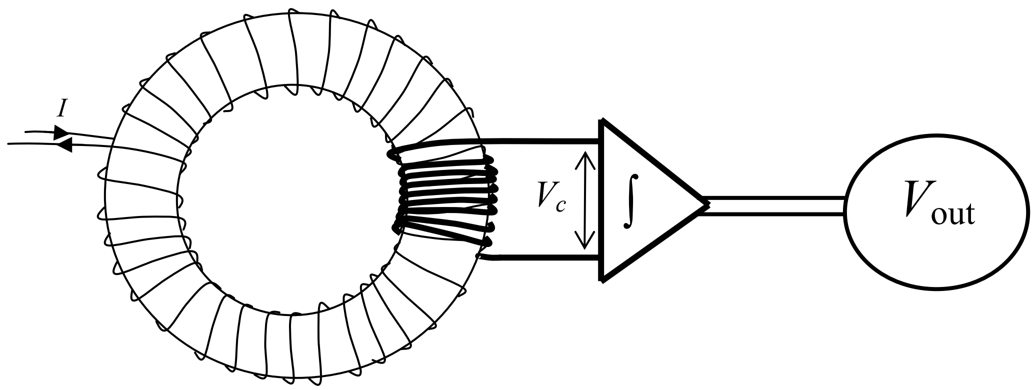


Derive a formula for the magnitude B of the magnetic induction between the plates of this capacitor at a distance of r mm from the centre. You may assume that $\mu_r = 1$. What is the shape of the **B** field? [6]

Sketch the variation of B with radius from r = 0 to r = 10 mm, firstly in plane P1 through the gap, secondly in plane P2 parallel to the plates but outside the capacitor and cutting through the wire, as shown. [4]

PLEASE TURN OVER

8. The drawing below shows a coil with N turns wound on a toroidal yoke of ferromagnetic material, major radius R, which can be magnetized by passing current I through the primary winding. The magnetic field intensity at any point within the yoke is given approximately by the expression $H = \dfrac{NI}{2\pi R}$. Explain how the integrating circuit, shown connected to a secondary winding with n_c turns, can be used as a fluxmeter to measure the hysteresis properties of the material. [6]



In a particular experiment with an initially unmagnetised sample the following successive measurements were made:

I (A)	0	+1	0	-1	+2	0	-2	+3	0	-3	+4	0	-4	+5	0	-5
V_{out}	0	+1	+0.9	-1	+2.5	+2.4	-2.5	+3	+2.9	-3	+3.2	+3.1	-3.2	+3.21	+3.1	-3.21

Knowing the calibration constants of the equipment, it can be deduced from these results that the saturation value of the magnetic field intensity H_{sat} is 1000 A/m and the saturation value of the magnetic induction B_{sat} is 1.6 T.

Sketch, on the graph paper provided, the minor and major hysteresis loops and the normal magnetization curve corresponding to these observations. Mark the point at which you think saturation occurs. Label the axes and use the values given for H_{sat} and B_{sat} to set the scales. [6]

From your graph, and/or from the tabulated numbers, estimate the values of the following properties of the ferromagnetic material:

The remanance B_r
The coercivity H_c
The effective relative permeability $\mu_{r\text{-eff}}$
The saturation magnetization M_s [8]

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9. The Fresnel relations for light reflected and refracted at a plane uniform dielectric surface can be written as

$$t_{\perp} = \frac{2\cos\alpha}{\cos\alpha + (n_2/n_1)\cos\alpha'}$$

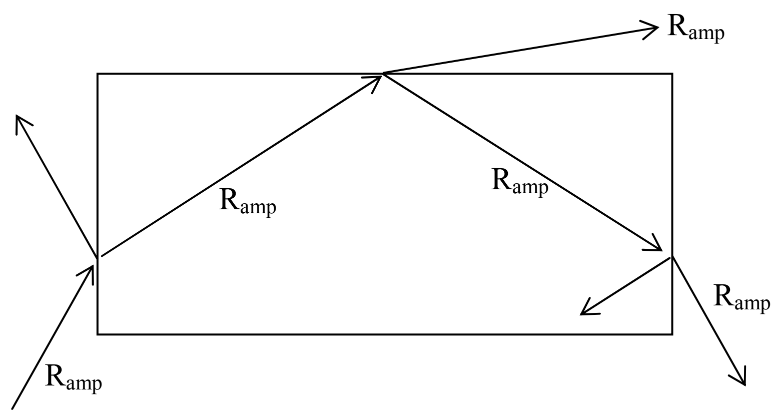
$$r_{\perp} = \frac{\cos\alpha - (n_2/n_1)\cos\alpha'}{\cos\alpha + (n_2/n_1)\cos\alpha'}$$

$$t_{\parallel} = \frac{2\cos\alpha}{(n_2/n_1)\cos\alpha + \cos\alpha'}$$

$$r_{\parallel} = \frac{(n_2/n_1)\cos\alpha - \cos\alpha'}{(n_2/n_1)\cos\alpha + \cos\alpha'}.$$

Draw a ray diagram and use it to define all of the quantities in these relations. [4]

The diagram below shows a ray of unpolarised light being refracted and reflected by a rectangular glass block with refractive index 1.5 in vacuum. If the angle of incidence on the first face is 45 degrees, what are the angles of reflection and refraction at the three faces for which segments of the ray are shown? Will the reflection at the second face be total? [5]



Work out the ratio of amplitudes of the two components of polarization for all of the ray segments marked "R_{amp}" in the diagram which have finite intensity. [11]

PLEASE TURN OVER

10. The dispersion relation for plane electromagnetic waves in a uniform conducting medium is

$$k^2 = \mu\varepsilon\omega^2\left(1+\frac{i\sigma}{\varepsilon\omega}\right).$$

Define all of the quantities in this equation. [3]

Explain what is meant by the skin depth δ in a good conductor, and use the dispersion relation to show that $\delta = \sqrt{\dfrac{2}{\mu\omega\sigma}}$ [7]

A submarine can detect radio transmissions when it places an antenna 50 cm below the surface of the sea. For electromagnetic waves falling normally on the surface, by what factor is the power of the received signal attenuated at

a) 198 kHz (Radio 4 long wave)
b) 93 MHz (Radio 4 FM),

compared with what would be received with the antenna immediately below the surface? [10]

[Take the conductivity of sea water as 5 (ohm m)$^{-1}$, the relative permittivity as 70 and the relative permeability as 1.]

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11. The four Maxwell equations in differential form are

$$\nabla.\mathbf{D} = \rho_f$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla.\mathbf{B} = 0$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

Name the observational law which each equation represents [3]

Explaining all necessary assumptions, use the appropriate Maxwell equations to derive the wave equation

$$\nabla^2 \mathbf{E} - \sigma\mu \frac{\partial \mathbf{E}}{\partial t} - \varepsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \qquad [5]$$

Give the simplified form of this equation for waves in vacuum. Show that a solution to the equation in vacuum is a plane wave of the form $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 \exp\{i(\omega t - \mathbf{k}.\mathbf{r} + \phi)\}$, and hence show that the phase velocity of electromagnetic waves in vacuum is

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}. \qquad [4]$$

Two plane waves, propagating in the same direction with the same frequency, have amplitude vectors $\mathbf{E}_{0x}$ and $\mathbf{E}_{0y}$ at right angles to one another, with phases ϕ_x and ϕ_y. Sketch how the real part of the sum of the wave vectors will vary on an x-y plot in the following cases:

$$E_{0x} = E_{0y}, \quad \phi_y - \phi_x = 0$$

$$E_{0x} = 2E_{0y}, \quad \phi_y - \phi_x = 180^0$$

$$E_{0x} = 2E_{0y}, \quad \phi_y - \phi_x = 90^0$$

$$E_{0x} = E_{0y}, \quad \phi_y - \phi_x = +45^0$$

State in each case whether the resultant polarization is plane, elliptical or circular. [8]

END OF PAPER